KAHANE'S CONSTRUCTION AND THE WEAK SEQUENTIAL COMPLETENESS OF L^1

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ABSTRACT. If λ is a normalized Lebesgue measure (arc length/ 2π) on $T = \{z \mid |z|=1\}$, the Gelfand map permits $L^\infty(\lambda)^*$ to be identified with $M(X)$, the space of finite Baire measures on X , the maximal ideal space of $L^\infty(\lambda)$. The measure m_0 in $M(X)$ represents λ : $\int_X f d m_0 = \int_T f d \lambda$ for all $f \in L^\infty(\lambda)$. Furthermore $\mu \ll m_0$ if and only if μ represents some measure of the form $\phi d\lambda$, $\phi \in L^1(\lambda)$. Using this fact and a sum constructed by J. P. Kahane, $\sum \hat{h}^{n(j)}$ when $\hat{h}$ is an appropriate function guaranteed by Urysohn's lemma, develops a proof that $L^1(\lambda)$ is weakly sequentially complete.

1. Introduction. If $T = \{z \mid |z|=1\}$ and λ is a normalized Lebesgue (arc length/ 2π) measure on T , then the Gelfand maps and the Riesz representation theorems permit the isometric and isomorphic identification of $L^\infty(\lambda)^*$ (the dual space of L^∞) with $M(X)$ the space of finite Baire measures on the maximal ideal space X of the Banach algebra L^∞ . The measure m_0 in $M(X)$ represents the linear functional

$$(1) \quad L(f) = \int_T f d\lambda = \int_X f d m_0, \quad f \in L^\infty.$$

It is natural to show that $L^1(\lambda)$ and $L^1(m_0)$ are the same space in different settings. They are twins in the following sense:

THEOREM 1. *If $\mu \in M(X)$ then $\mu \ll m_0$ if, and only if, there is a function ϕ in $L^1(\lambda)$ such that*

$$\int_X f d\mu = \int_T f \phi d\lambda, \quad f \in L^\infty.$$

In this case $\|\mu\|_{M(X)} = \|\phi\|_{L^1(\lambda)}$.

The author's initial proof of Theorem 1 used the fact that $L^1(\lambda)$ is weakly sequentially complete. However, Theorem 1 may be proved independently of this well-known theorem and used with a technique of J. P. Kahane to, in fact, prove it, in turn.

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THEOREM 2. (L^1 is weakly sequentially complete.) If $\{\phi_n \in L^1(\lambda) | n = 1, 2, \dots\}$ and $\lim_{n \rightarrow \infty} \int f \phi_n d\lambda$ exists for each f in $L^\infty(\lambda)$, then there is a $\phi \in L^1(\lambda)$ with

$$\lim_n \int f \phi_n d\lambda = \int f \phi d\lambda.$$

2. Main results. First let us recall some things about the Gelfand maps which identify

$$(2) \quad L^\infty(\lambda) \rightarrow C(X)$$

and

$$(3) \quad L^\infty(\lambda)^* \rightarrow C(X)^* \cong M(X),$$

when X is the maximal ideal space of $L^\infty(\lambda)$. The first map, $f \rightarrow \hat{f}$, is an isometric ring isomorphism which preserves complex conjugation [2, p. 170]. The dual map in (3), $L \rightarrow \hat{L}$, is defined by $\hat{L}(\hat{g}) = L(g)$, $g \in L^\infty$. The Riesz representation theorem identifies $C(X)^*$ with $M(X)$. The measure m_0 in $M(X)$ represents λ as in (1):

$$\int_X f dm_0 = \int_T f d\lambda, \quad f \in L^\infty.$$

THEOREM 1. A linear functional F in $L^\infty(\lambda)^*$ is represented by a measure μ in $M(X)$ which is absolutely continuous with respect to m_0 if and only if there is a function ϕ in $L^1(\lambda)$ with

$$F(g) = \int_X \hat{g} d\mu = \int_T g \phi d\lambda, \quad g \in L^\infty.$$

In this case $\|\mu\|_X = \|\phi\|_1$.

PROOF. First suppose μ is a measure in $M(X)$ with the Lebesgue decomposition $d\mu = C_E dm_0$ where C_E is the characteristic function of a closed G_δ set $E \subset X$. The sets of the form $U = \{x \in X | \hat{C}_K(x) = 1, K \subset T \text{ is } \lambda\text{-measurable}\}$ form a basis for the topology of X [2, pp. 169-170]. This together with Urysohn's lemma insures the existence of a sequence $\hat{u}_n \in C(X)$ with $\lim \hat{u}_n = C_E$. By (1) and the bounded convergence theorem for all h in L^∞ ,

$$(4) \quad \lim \int_T h u_n d\lambda = \lim \int_X \hat{h} \hat{u}_n dm_0 = \int_X \hat{h} d\mu.$$

Since $f \rightarrow \hat{f}$ is a ring and lattice isomorphism

$$\int_T |u_n - u_m|^2 d\lambda = \int_X |\hat{u}_n - \hat{u}_m|^2 dm_0.$$

The completeness of $L^2(\lambda)$ lets us choose a subsequence and a ϕ in L^2 with $\lim u_{n_j} = \phi$ a.e.- λ . Now the bounded convergence theorem shows that the left-hand side of (4) converges.

$$(5) \quad \int_T h \phi \, d\lambda = \int_X \hat{h} \, d\mu, \quad h \in L^\infty.$$

Consider the set

$$M = \text{span}\{C_E \, dm_0 \mid E \text{ (closed } G_\delta) \subset X\} \subset M(X).$$

If j is the natural embedding of $L^1(\lambda)$ into its second dual $L^\infty(\lambda)^*$ [1, p. 66], we have shown that the image of $j(L^1(\lambda))$ under the dual Gelfand map (3) contains M . Hence it contains $L^1(m_0)$ since j is another isometric isomorphism and M is a dense subset of $L^1(m_0)$ (the latter to be thought of as those measures in $M(X)$ absolutely continuous with respect to m_0).

(Observe at this point that if the weak sequential completeness of $L^1(\lambda)$ is assumed true, (4) allows us to assert directly that $\phi \in L^1(\lambda)$ exists with (5) holding.)

Since, by (5),

$$\left| \int_T f \phi \, d\lambda \right| = \left| \int_X \hat{f} \, d\mu \right|, \quad f \in L^\infty(\lambda),$$

taking the supremum over those f with $\|f\|_\infty \leq 1$ shows $\|\phi\|_1 = \|\mu\|_X$.

Conversely let $\phi \in L^1(\lambda)$ and $\mu \in M(X)$ represent F_ϕ :

$$(6) \quad F_\phi(g) = \int_T g \phi \, d\lambda = \int_X \hat{g} \, d\mu, \quad g \in L^\infty.$$

Let $E \subset X$ be a closed G_δ set with $m_0(E) = 0$. We shall show $\mu(E) = 0$, and thus $\mu \ll m_0$.

There is a sequence $\{\hat{u}_n \in C(X) \mid n = 1, 2, \dots\}$ with $0 \leq \hat{u}_n \leq 1$ and $\lim u_n = C_E$. Since $\hat{u}_n \geq 0$, $u_n \geq 0$ a.e.- λ . From (1),

$$0 = m_0(E) = \lim \int_X \hat{u}_n \, dm_0 = \lim \int_T |u_n| \, d\lambda.$$

Thus u_n converges to zero in $L^1(\lambda)$. Taking a subsequence suppose u_n converges to zero a.e.- λ .

By the Lebesgue dominated convergence theorem and (6) we see

$$0 = \lim \int_T u_n \phi \, d\lambda = \lim \int_X \hat{u}_n \, d\mu = \mu(E).$$

This ends the proof.

THEOREM 2. $L^1(\lambda)$ is weakly sequentially complete.

PROOF. Let $\{\phi_n \in L^1(\lambda) | n=1, 2, \dots\}$ have the property that

$$L(f) = \lim_n \int_T f \phi_n d\lambda$$

exists for each f in $L^\infty(\lambda)$. Let μ be that measure in $M(X)$ which represents L ; i.e. $\int_X f d\mu = L(f)$. That such a measure μ exists and is unique is a consequence of the principle of uniform boundedness and our identification of $L^\infty(\lambda)^*$ and $M(X)$.

If we show that μ is absolutely continuous with respect to m_0 then Theorem 1 says there is a ϕ in $L^1(\lambda)$ such that

$$L(f) = \lim \int_T f \phi_n d\lambda = \int_T f \phi d\lambda$$

and we are done.

Let $d\mu = g dm_0 + d\beta$ be the Lebesgue decomposition of μ with respect to m_0 with $\beta \perp m_0$. Suppose $\beta \neq 0$; i.e. that there is a closed G_δ set $E \subset X$ with $m_0(E) = 0$ and $|\int_E d\beta| = |\int_E d\mu| = a \neq 0$.

By a standard application of Urysohn's lemma there is a function h in $C(X)$ with $0 \leq h \leq 1$, $h(E) = 1$ and $h < 1$ off E . Taking powers of h , we may construct a sequence $\{\hat{h}_n = h^{k(n)} | n=1, 2, \dots\}$ so that for every strictly increasing sequence $\{n(j) | j=1, 2, \dots\}$ of positive integers the series

$$(7) \quad s = \sum_{j=1}^{\infty} (-1)^j \hat{h}_{n(j)}$$

has uniformly bounded partial sums and converges uniformly on compact subsets of $X \setminus E$ [3].

Now Kahane's construction requires:

$$(a) \quad \lim_n \left| \int \hat{h}_n d\mu \right| = a \neq 0.$$

$$(b) \quad \lim_j \int \hat{h}_n(\phi_j d\lambda) = \lim_j \int \hat{h}_n \hat{\phi}_j dm_0 = \int_X \hat{h}_n d\mu, \quad n = 1, 2, \dots$$

$$(c) \quad \lim_n \int_X \hat{h}_n \hat{\phi}_j dm_0 = 0, \quad j = 1, 2, \dots$$

(d) There is a function f in L^∞ with $f = s$ a.e.- m_0 and hence $\lim_{j \rightarrow \infty} \int_X s \hat{\phi}_j dm_0$ exists.

In (b), (c), and (d) the functions $\hat{\phi}_j \in L^1(m_0)$ are guaranteed by Theorem 1. To see (d), observe that the sequence of partial sums

$$\left\{ \mathcal{S}_n = \sum_{p=1}^n (-1)^p \hat{h}_{n(p)} \right\}$$

converges to s in $L^2(m_0)$. The complex conjugate preserving Gelfand isomorphism and (1) yield

$$\int_X |\hat{s}_n - \hat{s}_m|^2 dm_0 = \int_T |s_n - s_m|^2 d\lambda.$$

Thus $\{s_n | n=1, 2, \dots\}$ is uniformly bounded and a Cauchy sequence in $L^2(\lambda)$. However, if $s_n \rightarrow f$ in $L^2(\lambda)$ then $s=f$ a.e.- m_0 since

$$\lim \int_T |s_n - f|^2 d\lambda = \lim \int_X |\hat{s}_n - \hat{f}|^2 dm_0 = \lim \int_X |\hat{s}_n - s|^2 dm_0 = 0.$$

The rest of the argument parallels that of Kahane in [3] to show $\beta=0$.

3. Comments. It is hardly surprising that Kahane's methods yield the weak sequential completeness of L^1 since his work was an effort to decide the question: "Is L^1/H^1 weakly sequentially complete?" This question has since been answered affirmatively by Michael C. Mooney [4].

The series forming s in (7) is a useful construction. A modification of this series is used in [5] to develop the existence of certain interpolation sets.

Theorem 1 extends to the case of an arbitrary σ -finite measure λ ; the arguments in [2] are valid without requiring the special properties of Lebesgue measure or $\{|z|=1\}$. However, Theorem 1 is dependent on the fact that $L^\infty=(L^1)^*$ and this identification may fail for a non σ -finite measure [6].

A standard proof of Theorem 2 reduces the case of an arbitrary measure to the case of a σ -finite measure [1, pp. 289–291].

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