n -NORMAL LATTICES

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ABSTRACT. An n -normal lattice is a distributive lattice with 0 such that each prime ideal contains at most n minimal prime ideals. A relatively n -normal lattice is a distributive lattice such that each bounded closed interval is an n -normal lattice.

The main results of this paper are:

- (1) a distributive lattice L with 0 is n -normal if and only if for any $x_0, x_1, \dots, x_n \in L$ such that $x_i \wedge x_j = 0$ for any $i \neq j$, $i, j = 0, \dots, n$, $(x_0]^* \vee (x_1]^* \vee \dots \vee (x_n]^* = L$,
- (2) a distributive lattice L is relatively n -normal if and only if for any $n+1$ incomparable prime ideals $P_0, P_1, \dots, P_n$, $P_0 \vee P_1 \vee \dots \vee P_n = L$.

1. Introduction. Grätzer and Schmidt characterized a Stone lattice as a distributive pseudocomplemented lattice in which each prime ideal contains a unique minimal prime ideal; a detailed discussion of their theorem is given in [1]. Motivated by this characterization of Stone lattices, the author [1] studied distributive lattices with 0 in which each prime ideal contains a unique minimal prime ideal under the name "normal lattices".

Recently K. B. Lee [4] determined all the varieties of distributive pseudocomplemented lattices. He also generalized Grätzer and Schmidt's theorem by proving that for $1 \leq n < \omega$ the n th variety consists of all lattices such that each prime ideal contains at most n minimal prime ideals.

Thus in this paper we characterize " n -normal lattices" by means of a first-order sentence and then go on to generalize some of the results of [1].

Throughout this paper all lattices are distributive and all prime ideals are proper.

2. Preliminaries. In this section we present the preliminary results on which our main theorems hinge.

Lemma 2.1. *Let P be a prime ideal in a lattice L with 0. Then $O(P) =$*

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$\{x \in L: x \wedge y = 0 \text{ for some } y \in L \setminus P\}$ is the intersection of all the minimal prime ideals of L which are contained in P .

Proof. See [1, Proposition 2.2]. $\square$

Lemma 2.2. *If L_1 is a sublattice of a lattice L and P_1 is a prime ideal in L_1 then there exists a prime ideal P in L such that $P_1 = L_1 \cap P$.*

Proof. This result is well known. For an explicit proof see [1, Lemma 3.4]. $\square$

The final lemma is crucial to this paper; it was suggested by a recent result of Hindman [3, Theorem 1.8].

Lemma 2.3. *Let J be an ideal in a lattice L . For a given positive integer $n \geq 2$, the following conditions are equivalent:*

- (i) *for any $x_1, x_2, \dots, x_n \in L$ which are "pairwise in J ", i.e. $x_i \wedge x_j \in J$ for any $i \neq j$, there exists k such that $x_k \in J$,*
- (ii) *for any ideals $J_1, J_2, \dots, J_n$ in L such that $J_i \cap J_j \subseteq J$ for any $i \neq j$, there exists k such that $J_k \subseteq J$,*
- (iii) *J is the intersection of at most $n - 1$ distinct prime ideals.*

Proof. (i) and (ii) are easily seen to be equivalent.

(iii) $\Rightarrow$ (i). Suppose $P_1, P_2, \dots, P_k$ are k ($1 \leq k \leq n - 1$) distinct prime ideals such that $J = P_1 \cap P_2 \cap \dots \cap P_k$.

Let $x_1, x_2, \dots, x_n$ be elements of L such that $x_i \wedge x_j \in J$ for all $i \neq j$. Suppose no element x_i is a member of J . Then, for each r ($1 \leq r \leq k$) there is at most one i ($1 \leq i \leq n$) such that $x_i \notin P_r$. Since $k < n$ there is some i such that $x_i \in P_1 \cap P_2 \cap \dots \cap P_k$.

(i) $\Rightarrow$ (iii). Let J be an ideal which satisfies condition (i). If J is a prime ideal, i.e. condition (i) actually holds when $n = 2$, then (iii) is trivially true. Thus we may assume that there is a largest integer $m < n$ such that condition (i) does not hold for J (consequently condition (i) holds for $m + 1, m + 2, \dots, n$). For some $m < n$ we may suppose that there exist elements $a_1, a_2, \dots, a_m \in L$ such that $a_i \wedge a_j \in J$ for $i \neq j, i = 1, \dots, m, j = 1, \dots, m$, yet $a_1, a_2, \dots, a_m \notin J$.

As L is a distributive lattice $J: a_i = \{x \in L: x \wedge a_i \in J\}$ is an ideal for any $i \in \{1, \dots, m\}$. Each $J: a_i$ is in fact a prime ideal. Firstly $J: a_i \neq L$ since $a_i \notin J$. Secondly, suppose that b and c are in L and $b \wedge c \in J: a_i$. Consider the set of $m + 1$ elements $\{a_1, a_2, \dots, a_{i-1}, b \wedge a_i, c \wedge a_i, a_{i+1}, \dots, a_m\}$. This set is pairwise in J and so, either $b \wedge a_i \in J$

or $c \wedge a_i \in J$ since condition (i) holds for $m+1$. That is, $b \in J: a_i$ or $c \in J: a_i$ and $J: a_i$ is prime.

Clearly $J \subseteq \bigcap_{1 \leq i \leq m} J: a_i$. If $w \in \bigcap_{1 \leq i \leq m} J: a_i$ then $w, a_1, \dots, a_m$ are pairwise in J and so $w \in J$. Hence $J = \bigcap_{1 \leq i \leq m} J: a_i$ is the intersection of $m < n$ prime ideals. $\square$

Definition 2.4. An ideal $J \neq L$ satisfying any of the equivalent conditions of Lemma 2.3 for some positive integer $n \geq 2$ is called an *n-prime ideal*.

Thus an ideal is a prime ideal if and only if it is a 2-prime ideal. It should also be noted that if $2 \leq n \leq m$ then any n -prime ideal is also an m -prime ideal.

3. Main theorems.

Definitions. 3.1. Let L be a lattice with 0. Then L is called *n-normal* for $n \geq 1$ if each prime ideal in L contains at most n minimal prime ideals.

3.2. Let L be a lattice with 0. Then L is called *sectionally n-normal* for $n \geq 1$ if each initial segment $[0, x]$, $x \in L$, is an n -normal lattice.

3.3. Let L be a lattice. Then L is called *relatively n-normal* for $n \geq 1$ if each closed interval $[x, y]$, $x \leq y$, $x, y \in L$, is an n -normal lattice.

Under the name of normal (resp. relatively normal), 1-normal (resp. relatively 1-normal) lattices have been extensively studied by the author in a previous paper [1]. We now extend some of the fundamental results of that paper to the case of any positive integer, n say.

Theorem 3.4. Let L be a lattice with 0 and let n be a positive integer. The following conditions are equivalent:

- (i) L is n -normal,
- (ii) for each prime ideal P , $O(P)$ is $(n+1)$ -prime,
- (iii) for any $n+1$ distinct minimal prime ideals $M_0, M_1, \dots, M_n$,

$$M_0 \vee M_1 \vee \dots \vee M_n = L,$$

- (iv) for any $x_0, x_1, \dots, x_n \in L$ such that $x_i \wedge x_j = 0$ for $i \neq j$, $i = 0, \dots, n$, $j = 0, \dots, n$,

$$(x_0]^* \vee (x_1]^* \vee \dots \vee (x_n]^* = L.$$

Proof. (i) and (iii) are equivalent since each proper ideal in a distributive lattice is contained in some prime ideal.

(i) $\Rightarrow$ (ii). This follows immediately from Lemmas 2.1 and 2.3.

(ii) $\Rightarrow$ (iv). Suppose that $x_0, x_1, \dots, x_n$ are $n+1$ elements of L such that $x_i \wedge x_j = 0$ for $i \neq j$. Suppose that $(x_0]^* \vee \dots \vee (x_n]^* \neq L$. Then there is a prime ideal P such that $(x_0]^* \vee \dots \vee (x_n]^* \subseteq P$. Hence, $x_0, x_1, \dots, x_n \in L \setminus O(P)$. But this contradicts (ii) since $x_i \wedge x_j = 0 \in O(P)$ for all $i \neq j$. Thus (ii) does imply (iv).

(ii) $\Rightarrow$ (i). By Lemma 2.3, (ii) implies that $O(P) = P_1 \cap \dots \cap P_k$ for some $k \leq n$ distinct prime ideals $P_1, \dots, P_k$. Let Q be a minimal prime ideal contained in P . By Lemma 2.1, $O(P) \subseteq Q$ so that $P_i \subseteq Q$ for some i ($1 \leq i \leq k$). As Q is minimal $Q = P_i$ and so there are at most k , and hence at most n , minimal prime ideals contained in P .

(iv) $\Rightarrow$ (ii). Suppose $a_0, a_1, \dots, a_n$ are $n+1$ elements of L which are pairwise in $O(P)$. Then, for any $i \neq j$, $a_i \wedge a_j \wedge y_{ij} = 0$ for some $y_{ij} \in L \setminus P$. Put $b_i = a_i \wedge y_{i1} \wedge y_{i2} \wedge \dots \wedge y_{ii-1} \wedge y_{ii+1} \wedge \dots \wedge y_{in}$ for each $i = 0, 1, \dots, n$. For distinct i and j , $b_i \wedge b_j = 0$. By (iv) $(b_0]^* \vee \dots \vee (b_n]^* = L$. Hence $(b_k]^* \not\subseteq P$ for some $k = 0, \dots, n$. Thus $b_k \in O(P)$. That is, $a_k \wedge y_{k1} \wedge \dots \wedge y_{kk-1} \wedge y_{kk+1} \wedge \dots \wedge y_{kn} \wedge x = b_k \wedge x = 0$ for some $x \in L \setminus P$. But $y_{k1}, \dots, y_{kk-1}, y_{kk+1}, \dots, y_{kn}$ are not in P and so $a_k \in O(P)$. $\square$

Corollary 3.5. *For any $n+1$ elements $x_0, x_1, \dots, x_n$ in an n -normal lattice L*

$$(x_0 \wedge \dots \wedge x_n]^* = \bigvee_{0 \leq i \leq n} (x_0 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_n]^*.$$

Proof. Let $b_i = x_0 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_n$ for each $0 \leq i \leq n$. Suppose that $x \in (x_0 \wedge \dots \wedge x_n]^*$. Then $x \wedge x_0 \wedge \dots \wedge x_n = 0$ so that, for $i \neq j$, $(x \wedge b_i) \wedge (x \wedge b_j) = 0$. From the previous theorem, $x \in (x \wedge b_0]^* \vee \dots \vee (x \wedge b_n]^*$ so that $x = a_0 \vee \dots \vee a_n$ for some $a_i \in L$ such that $a_i \wedge x \wedge b_i = 0$. Then $x = (a_0 \wedge x) \vee \dots \vee (a_n \wedge x)$ and $a_i \wedge x \in (b_i]^*$ and so

$$(x_0 \wedge \dots \wedge x_n]^* \subseteq \bigvee_{0 \leq i \leq n} (x_0 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_n]^*.$$

The reverse inclusion is trivial. $\square$

Remark. Grätzer and Lakser (see [2, Lemma 2, p. 167]) have shown that condition (iv) of Theorem 3.4 is equivalent to Lee's condition which characterizes the n th variety, for $0 < n < \omega$, of distributive pseudocomplemented lattices. Thus, Theorem 3.4 should be compared with Lee's Theorem 2 of [4].

Theorem 3.6. *The following conditions are equivalent for a lattice L with 0 :*

- (i) L is sectionally n -normal,
- (ii) L is n -normal,
- (iii) each ideal J in L is an n -normal sublattice.

Proof. (iii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). Let J be an ideal in L and suppose $x_0, x_1, \dots, x_n \in J$ are such that $x_i \wedge x_j = 0$ for all $i \neq j$. Let $(x_i)^+ = \{y \in J: y \wedge x_i = 0\}$. Note that $(x_i)^+ = (x_i)^* \cap J$. By Theorem 3.4, $(x_0)^* \vee \dots \vee (x_n)^* = L$ and so $J = J \cap L = (J \cap (x_0)^*) \vee \dots \vee (J \cap (x_n)^*) = (x_0)^+ \vee \dots \vee (x_n)^+$, since the lattice of ideals of a distributive lattice is distributive.

(i) $\Rightarrow$ (ii). Suppose that (i) holds and $x_0, x_1, \dots, x_n \in L$ are such that $x_i \wedge x_j = 0$ for $i \neq j$. Let $w \in L$ be arbitrary. Then $w, x_0, x_1, \dots, x_n \in [0, w \vee x_0 \vee \dots \vee x_n] = I$. As I is n -normal, $w \in (x_0)^+ \vee \dots \vee (x_n)^+$ where $(x_i)^+ = \{y \in I: y \wedge x_i = 0\}$. Hence $w \in (x_0)^* \vee \dots \vee (x_n)^*$, i.e. $L = (x_0)^* \vee \dots \vee (x_n)^*$. $\square$

Theorem 3.7. *Let L be a lattice and let n be a positive integer. The following conditions are equivalent:*

- (i) L is relatively n -normal,
- (ii) there is no homomorphism from L onto $\bar{B}_{n+1}$, where $\bar{B}_{n+1}$ is the lattice obtained by adding a new largest element to the Boolean algebra B_{n+1} of subsets of $\{0, 1, 2, \dots, n\}$,
- (iii) for any $n+1$ pairwise incomparable prime ideals $P_0, P_1, \dots, P_n$,

$$P_0 \vee P_1 \vee \dots \vee P_n = L.$$

Proof. (i) $\Rightarrow$ (ii). Suppose there is a homomorphism f from L onto $\bar{B}_{n+1}$. Let $a \in f^*(1)$. For each k in $\{0, 1, \dots, n\}$ let $x_k \in f^*(\{0, \dots, k-1, k+1, \dots, n\})$. Consider $I = [x_0 \wedge \dots \wedge x_n, x_0 \vee \dots \vee x_n \vee a]$. For each i in $\{0, 1, \dots, n\}$ let $b_i = x_0 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_n$. Then, if $i \neq j$, $b_i \wedge b_j = x_0 \wedge \dots \wedge x_n$, the zero element of I . By Theorem 3.4, since $a \leq x_0 \vee \dots \vee x_n \vee a$, (i) implies that $a \leq y_0 \vee \dots \vee y_n$ for some $\{y_0, \dots, y_n\} \subseteq I$ such that $y_i \wedge b_i = x_0 \wedge \dots \wedge x_n$ for each i .

But $f(x_0 \wedge \dots \wedge x_n) = \emptyset$ so, for each i , $f(y_i) \wedge f(b_i) = \emptyset$ while $f(b_i) = \{i\}$. Thus, for each i , $f(y_i) \in B_{n+1}$, so $f(a) \in B_{n+1}$, a contradiction.

(ii) $\Rightarrow$ (iii). Suppose that (iii) does not hold. Let $P_0, \dots, P_n$ be

$n + 1$ pairwise incomparable prime ideals such that $P_0 \vee \dots \vee P_n \neq L$. Let P be any prime ideal such that $P_0 \vee \dots \vee P_n \subseteq P$.

Define f mapping L to $\bar{B}_{n+1}$ by the rule $f(x) = \{k \in \{0, 1, \dots, n\} : x \notin P_k\}$ if $x \in P$, and $f(x) = 1$ if $x \notin P$. It is easily checked that f is a homomorphism. That $f^*(1)$, $f^*(\emptyset)$, and $f^*(\{0, 1, \dots, n\})$ are nonempty is clear. Let F be a proper nonempty subset of $\{0, 1, \dots, n\}$. For each pair of distinct elements i and j of $\{0, 1, \dots, n\}$ let $x_{ij} \in P_i \setminus P_j$ and let $x = \bigwedge_{i \notin F} (\bigvee_{j \in F} x_{ij})$. Then $f(x) = F$. Consequently f is onto $\bar{B}_{n+1}$ and (iii) fails to hold.

(iii) $\Rightarrow$ (i). Let $I = [x, y]$ be an interval in L and suppose that $M_0, M_1, \dots, M_n$ are $n + 1$ distinct minimal prime ideals in I . By Lemma 2.2 there are prime ideals $P_0, P_1, \dots, P_n$ in L such that $M_i = I \cap P_i$ for all $i = 0, \dots, n$. Clearly the P_i are pairwise incomparable and so (ii) implies that $P_0 \vee \dots \vee P_n = L$.

Let $a \in I$. Then $a = c_0 \vee \dots \vee c_n$ for suitable $c_i \in P_i$. Now $c_i \leq a \leq y$ so that $c_i \vee x \in I$ for all i . As $x \in M_i, x \in P_i$ for each i and hence $c_i \vee x \in I \cap P_i = M_i$. Then $a = a \vee x = (c_0 \vee x) \vee \dots \vee (c_n \vee x) \in M_0 \vee \dots \vee M_n$ and so $M_0 \vee \dots \vee M_n = I$. Theorem 3.4 now shows that (i) holds. $\square$

Of course, if L has a smallest element and is relatively n -normal then L is n -normal. The following result demonstrates the failure of the converse.

Theorem 3.8. *For each n ($0 < n < \omega$), there exists a lattice which is 1-normal (and hence n -normal) but is not relatively n -normal.*

Proof. Let n be given. Let L be the sublattice of the power set of $\{0, 1, \dots, n + 3\}$ which is generated by the sets $\{0, 1\}, \{0, 2\}, \dots, \{0, n + 1\}, \{0, 1, 2, \dots, n + 1, n + 2\}$ and $\{n + 3\}$. Then the interval $[\{0\}, \{0, 1, 2, \dots, n + 1, n + 2\}]$ is not n -normal since it is isomorphic to $\bar{B}_{n+1}$. Therefore L is not relatively n -normal.

The only minimal prime ideals in L are $(\{n + 3\})$ and $(\{0, 1, \dots, n + 2\})$ and they are comaximal. Thus L is 1-normal. $\square$

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