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λ CONNECTIVITY AND MAPPINGS ONTO A CHAINABLE INDECOMPOSABLE CONTINUUM

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ABSTRACT. A continuum (i.e., a compact connected nondegenerate metric space) M is said to be λ connected if any two of its points can be joined by a hereditarily decomposable continuum in M. Here we prove that a plane continuum is λ connected if and only if it cannot be mapped continuously onto Knaster's chainable indecomposable continuum with one endpoint. Recent results involving aposyndesis and decompositions to a simple closed curve are extended to λ connected continua.

Throughout this paper D will denote Knaster's chainable indecomposable continuum with one endpoint (see [7, p. 332] or [9, Example 1, p. 205]), I will denote the unit interval, and h will denote the function of I onto itself defined by $h(t) = 2t$ for $t \leq \frac{1}{2}$ and $h(t) = 2 - 2t$ for $t > \frac{1}{2}$. D can be represented as an inverse limit of unit intervals, indexed by the positive integers, where the bonding map between successive terms is always h.

In [10], J. W. Rogers, Jr. proved that every indecomposable continuum can be mapped continuously onto D. Recently D. P. Bellamy [1] generalized this theorem by showing that D is a continuous image of each indecomposable compact connected nondegenerate Hausdorff space. Our principal tool (presented in the following theorem) is derived from Bellamy's proof.

Theorem 1. *Suppose that M is a continuum and $\{G_n\}_{n=1}^{\infty}$ is a sequence of nonempty open sets in M such that* (1) *the closures of G_1 and G_2 are disjoint,* (2) *for each n, $G_{2n+1} \cup G_{2n+2} \subset G_{2n-1}$, and* (3) *for each n, there is a separation $A_n \cup B_n$ of $M - G_{2n}$ such that $G_{2n+1} \subset A_n$ and $G_{2n+2} \subset B_n$. Then M can be mapped continuously onto D.*

Proof. Following Bellamy [1, Theorem (proof)], we let f_1 be a Urysohn

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function of M onto I such that $f_1(G_1) = 0$ and $f_1(G_2) = 1$. Proceeding inductively, we suppose a continuous function f_i of M onto I has been defined for each positive integer $i \leq n$ such that for each $i > 1$, $h \circ f_i = f_{i-1}$ and such that for each i, $f_i(G_{2i-1}) = 0$ and $f_i(G_{2i}) = 1$. Define the function f_{n+1} of M onto I by

$$\begin{aligned} f_{n+1}(x) &= \tfrac{1}{2} f_n(x) && \text{if } x \in A_n, \\ &= 1 - \tfrac{1}{2} f_n(x) && \text{if } x \in B_n, \\ &= \tfrac{1}{2} && \text{if } x \in G_{2n}. \end{aligned}$$

Note that f_{n+1} is a continuous function, $h \circ f_{n+1} = f_n$, $f_{n+1}(G_{2n+1}) = 0$, and $f_{n+1}(G_{2n+2}) = 1$. The sequence $\{f_n\}_{n=1}^{\infty}$ induces a continuous function of M onto D.

Theorem 2. *A plane continuum M is λ connected if and only if M cannot be mapped continuously onto D.*

Proof. Suppose that M is λ connected. In [6], it is proved that every planar continuous image of a λ connected continuum is λ connected. Note that since D is chainable and indecomposable, it is planar [2, Theorem 4] and not λ connected [9, Theorem 7, p. 212]. It follows that M cannot be mapped continuously onto D.

To establish the sufficiency part of this theorem we assume that M is not λ connected. According to [6, Theorems 1 and 3], there exists an indecomposable continuum Y in M such that every subcontinuum of M that contains a nonempty open subset of Y contains Y.

Let G_1 and G_2 be open subsets of M that have disjoint closures such that $Y \cap G_1 \neq \emptyset \neq Y \cap G_2$. Proceeding inductively, we assume that open subsets G_{2i-1} and G_{2i} of M have been defined for $1 \leq i \leq n$ such that (1) $G_{2i-1} \cap Y \neq \emptyset \neq G_{2i} \cap Y$, (2) for each $i < n$, $G_{2i+1} \cup G_{2i+2} \subset G_{2i-1}$, and (3) for each $i < n$, there is a separation $A_i \cup B_i$ of $M - G_{2i}$ such that $G_{2i+1} \subset A_i$ and $G_{2i+2} \subset B_i$. Since every subcontinuum of M that contains $G_{2n-1} \cap Y$ contains $G_{2n} \cap Y$, there exist distinct components W and Z of $M - G_{2n}$ such that $W \cap G_{2n-1} \cap Y \neq \emptyset \neq Z \cap G_{2n-1} \cap Y$. Hence there exists a separation $A_n \cup B_n$ of $M - G_{2n}$ such that $Y \cap G_{2n-1} \cap A_n \neq \emptyset \neq Y \cap G_{2n-1} \cap B_n$ [9, Theorem 2, p. 169]. Define G_{2n+1} and G_{2n+2} to be open subsets of M in $G_{2n-1} \cap A_n$ and $G_{2n-1} \cap B_n$, respectively, such that $G_{2n+1} \cap Y \neq \emptyset \neq G_{2n+2} \cap Y$.

Since the sequence $\{G_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 1, it follows that D is a continuous image of M.

Definition. For each set A in a continuum M, let $K(A)$ be the intersection of the collection consisting of every continuum in M that contains A in its interior relative to M.

This concept is introduced by F. B. Jones [8, Theorem 2]. There the K function is restricted to points (rather than subsets) of a continuum.

Theorem 3. *A plane continuum M can be mapped continuously onto D if and only if for some point x of M, the set $K(x)$ contains an indecomposable continuum.*

Proof. It is known that M is λ connected if and only if for each point x of M, every continuum in the set $K(x)$ is decomposable [5, Theorem 5]. Hence this theorem follows directly from Theorem 2.

In [11], H. E. Schlais establishes the following:

Theorem. *If M is a hereditarily decomposable continuum, then for each point x of M, the interior of $K(x)$ relative to M is void.*

E. J. Vought in [12] points out that Schlais' argument [11, Theorem 9 (proof)] also indicates that for each continuum of condensation H in a hereditarily decomposable continuum M, the interior of $K(H)$ relative to M is void. Using this fact, Vought then proves the following:

Theorem. *Suppose that M is a hereditarily decomposable continuum that is not separated by any of its subcontinua. Then M has a monotone upper semicontinuous decomposition each of whose elements has void interio and whose quotient space is a simple closed curve.*

In [4] the author extends Schlais' theorem to λ connected plane continua. However, it follows from [3, Theorem 2] and [5, Theorem 1] that every λ connected plane continuum that is not separated by any of its subcontinua is hereditarily decomposable. Hence [4] cannot be used to generalize Vought's decomposition theorem.

The following result extends Schlais' theorem to all λ connected continua, and leads us to a generalization of Vought's decomposition theorem.

Theorem 4. *If M is a λ connected continuum, then for each connected nowhere dense subset H of M, the interior of $K(H)$ relative to M is void.*

Proof. Assume there exists a connected nowhere dense subset H of M such that the interior of $K(H)$ relative to M is not empty. Let U be a nonempty open subset of M whose closure is contained in the interior of $K(H) - H$. Define L to be the component of $M - U$ that contains H. Let Z denote the intersection of L and the boundary of U. Define G_1 and G_2 to be nonempty open subsets of M such that $Z \subset G_1$, $G_2 \subset U$, and the closures of G_1 and G_2 are disjoint subsets of $K(H) - H$. We proceed inductively. Assume that open subsets G_{2i-1} and G_{2i} of M have been defined for $1 \leq i \leq n$ such that (1) $Z \subset G_{2i-1}$, (2) L and the closure of G_{2i} are disjoint, (3) for each $i < n$, $G_{2i+1} \cup G_{2i+2} \subset G_{2i-1}$, and (4) for each $i < n$, there is a separation $A_i \cup B_i$ of $M - G_{2i}$ such that $G_{2i+1} \subset A_i$ and $G_{2i+2} \subset B_i$.

Note that since L and the closure of G_{2n} are disjoint, P = (the component of $M - G_{2n}$ that contains H) contains Z.

Suppose that P contains G_{2n-1}. Since H does not lie in the interior of P, there exists a sequence of points $\{x_i\}_{i=1}^{\infty}$ in $M - (P \cup G_{2n})$ converging to a point of H. For each positive integer i, define X_i to be the x_i-component of $M - G_{2n}$. The limit superior X of $\{X_i\}_{i=1}^{\infty}$ is a continuum in M that meets both H and the boundary of G_{2n} [7, Theorem 2-101, p. 101]. Since $G_{2n-1} \subset P$, for each positive integer i, $G_{2n-1} \cap X_i = \emptyset$, which implies that $G_{2n-1} \cap X = \emptyset$. Since $Z \subset G_{2n-1}$ and $X \cap L \neq \emptyset$, the continuum X does not meet U. But this implies that X is a subset of L, which contradicts the assumption that L does not meet the closure of G_{2n}. Hence there exists a component Q of $M - G_{2n}$, distinct from P, that intersects G_{2n-1}.

Define $A_n \cup B_n$ to be a separation of $M - G_{2n}$ such that $P \subset A_n$ and $Q \subset B_n$. Let G_{2n+1} be an open set in $A_n \cap G_{2n-1}$ that contains Z. Let G_{2n+2} be a nonempty open set in $B_n \cap G_{2n-1}$. Note that since $L \subset A_n$ and $G_{2n+2} \subset B_n$, L and the closure of G_{2n+2} are disjoint.

The sequence $\{G_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 1. Hence the indecomposable plane continuum D is a continuous image of the λ connected continuum M, which is impossible [6, Theorem 5]. It follows that for each nowhere dense connected subset H of M, the interior of $K(H)$ relative to M is void.

Theorem 5. *Suppose that M is a λ connected continuum that is not separated by any of its subcontinua. Then M has a monotone upper semi-continuous decomposition each of whose elements has void interior and whose quotient space is a simple closed curve.*

Proof. This theorem follows directly from Theorem 4 and [12, Theorem 2 (proof)].

To see that Theorem 5 actually is a generalization of Vought's theorem, consider the continuum M that is the union of a disk T and a ray that limits on T and has only its endpoint in T. The λ connected continuum M is not separated by any of its subcontinua. Since M contains a disk, it is not hereditarily decomposable.

REFERENCES

1. D. P. Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach*, Pacific J. Math. 47 (1973), 303–309.

2. R. H. Bing, *Snake-like continua*, Duke Math. J. **18** (1951), 653–663. MR 13, 265.

3. C. L. Hagopian, *Planar images of decomposable continua*, Pacific J. Math. 42 (1972), 329–331.

4. ———, *Schlais' theorem extends to λ connected plane continua*, Proc. Amer. Math. Soc. 40 (1973), 265–267.

5. ———, *Characterizations of λ connected plane continua*, Pacific J. Math. 49 (1973).

6. ———, *λ connected plane continua*, Trans. Amer. Math. Soc. 191 (1974), 277–287.

7. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961. MR 23 #A2857.

8. F. B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math. 70 (1948), 403–413. MR 9, 606.

9. K. Kuratowski, *Topology*, Vol. 2, 3rd ed., Monografie Mat., Tom 21, PWN, Warsaw, 1961; English transl., Academic Press, New York; PWN, Warsaw, 1968. MR 24 #A2958; 41 #4467.

10. J. W. Rogers, Jr., *On mapping indecomposable continua onto certain chainable indecomposable continua*, Proc. Amer. Math. Soc. **25** (1970), 449–456. MR 41 #1017.

11. H. E. Schlais, *Non-aposyndesis and non-hereditary decomposability*, Pacific J. Math. 45 (1973), 643–652.

12. E. J. Vought, *Monotone decompositions of continua not separated by any subcontinua*, Trans. Amer. Math. Soc. 192 (1974), 67–78.

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