COEFFICIENTS ASSOCIATED WITH THE EXPANSION OF CERTAIN PRODUCTS

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ABSTRACT. Simple combinatorial interpretations are given for the coefficients of the polynomials $H_n(x, y)$ and $G_n(x, y)$ defined by $\prod(1+x^n y^m t) = \sum G_n(x, y) t^{n/(x)_n (y)_n}$ and $\prod(1-x^n y^m t)^{-1} = \sum H_n(x, y) t^{n/(x)_n (y)_n}$.

1. Introduction. Consider the products

$$(1.1) \quad \prod_{m,n=0}^{\infty} (1 + x^n y^m t) = \sum_{n=0}^{\infty} G_n(x, y) \frac{t^n}{(x)_n (y)_n}$$

and

$$(1.2) \quad \prod_{m,n=0}^{\infty} (1 - x^n y^m t)^{-1} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{(x)_n (y)_n}$$

where

$$(x)_n = (1-x)(1-x^2) \dots (1-x^n), \quad (x)_0 = 1.$$

The products (1.1) and (1.2) were first considered by Carlitz [2] and Wright [7] in response to a research problem posed by Bellman [1]. They show that $G_n(x, y)$ and $H_n(x, y)$ are polynomials in x and y with integral coefficients of degree $\frac{1}{2}n(n-1)$ in each variable and that

$$(1.3) \quad H_n(x, y) = y^{n(n-1)/2} G_n(x, 1/y).$$

Wright conjectured [7], [8] that the coefficients of these polynomials are non-negative integers. This conjecture was settled by Gordon [4] who, using a combinatorial argument, proved this conjecture and the analogous result for the more general product

$$(1.4) \quad \prod_{k=1}^{\infty} (1 - x_1^{k_1} \dots x_s^{k_s} t)^{-1}.$$

Here we supply a simple combinatorial meaning for the coefficients of

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the polynomials $H_n(x, y)$ and $G_n(x, y)$. Let $\pi = (i_1, i_2, \dots, i_n)$ be a permutation of $Z_n = \{1, 2, \dots, n\}$. We say that π has r rises and s readings if there are $r - 1$ subscripts j such that $i_j < i_{j+1}$ and if it is necessary to make s left to right scans of π in order that the elements are arranged in the order $1, 2, \dots, n$. Thus, for example, the permutation $\pi = (216354)$ has 3 rises and 4 readings. The number, $R_n(r, s)$, of permutations of Z_n with r rises and s readings has been studied [3] and suggests the concepts of *weighted rises* and *weighted readings* which are of interest here.

We say that the permutation $\pi = (i_1, i_2, \dots, i_n)$ has u *weighted rises* and v *weighted readings* if each rise position $i_j < i_{j+1}$ is assigned the weight $n - j$ and, if the element i is read on the j th scan, then it is assigned the weight $s_i = j - 1$ and

$$\sum_{i_j < i_{j+1}} (n - j) = u, \quad \sum_{i=1}^n s_i = v.$$

Thus, continuing the earlier example, the permutation $\pi = (216354)$ has $u = 4 + 2 = 6$ weighted rises and $v = 0 \cdot 1 + 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 1 = 8$ weighted readings. We remark that a permutation π has v weighted readings implies that π^{-1} has $\binom{n}{2} - v$ weighted rises.

Let $c_n(u, v)$ denote the number of permutations of Z_n with u weighted rises and v weighted readings. The main result of this paper is to show that

$$(1.5) \quad G_n(x, y) = \sum_{u, v=0}^{\binom{n}{2}} c_n(u, v) x^u y^v.$$

It follows at once from (1.3), (1.5) and our earlier remark that

$$(1.6) \quad H_n(x, y) = \sum_{u, v=0}^{\binom{n}{2}} d_n(u, v) x^u y^v,$$

where $d_n(u, v)$ denotes the number of permutations of Z_n with u weighted rises whose inverses have v weighted rises.

It is possible to give an alternative combinatorial description of these coefficients. Namely, $d_n(u, v)$ is equal to the number of pairs $\{(a_1, a_2, \dots, a_n), (b_1, \dots, b_n)\}$ of n -tuples of positive integers satisfying

(i) for each $i \geq 1$ and $j \in \{1, \dots, n\}$, the number of occurrences of i among $a_1, a_2, \dots, a_j$ (respectively, $b_1, b_2, \dots, b_j$) is at least as large as the number of occurrences of $i + 1$ among $a_1, a_2, \dots, a_j$ (respectively, $b_1, b_2, \dots, b_j$);

(ii) for each i , the number of a_j 's which are equal to i is the same as the number of b_k 's which are equal to i ;

(iii) $\sum \{i | a_i < a_{i+1}\} = u$, $\sum \{j | b_j < b_{j+1}\} = v$.

For example, $d_5(4, 5) = 4$ corresponds to the four pairs

$$(12121), (12112); \quad (12131), (12113); \quad (12131), (11231); \quad (12132), (11232).$$

The equivalence of these two interpretations can be established directly, but it is of interest to note that the second follows from some recent work of Stanley in the theory of plane partitions. A sketch of the details involved begins by noting that if $g_i(x)$ is the polynomial defined by (7) [6], then $H_n(x, x) = g_n(x)$. Next, a generalization of Theorem 5.1 of [6] shows that

$$x^n y^n H_n(x, y) = (x)_n (y)_n \sum_{\lambda} F(\lambda, x) F(\lambda, y).$$

Applying the result of Corollary 7.2 of [5], we are able to deduce the second interpretation for the coefficients $d_n(u, v)$.

The author is indebted to the referee for pointing out the second combinatorial interpretation.

2. The main result. It follows from (1.1) that

$$(2.1) \quad \sum_{n=0}^{\infty} G_n(x, y) \frac{t^n}{(x)_n (y)_n} = \sum_{n, k, m=0}^{\infty} t^n x^k y^m N_n(k, m),$$

where $N_n(k, m)$ denotes the number of partitions of the bipartite (k, m) of the form

$$k = a_1 + \cdots + a_n, \quad m = b_1 + \cdots + b_n,$$

where the a_i and b_i are nonnegative integers subject to the conditions $a_1 \leq a_2 \leq \cdots \leq a_n$ and if $a_i = a_{i+1}$ then $b_i \neq b_{i+1}$. We can then arrange the parts of the partition so that if $a_i = a_{i+1}$ then $b_i > b_{i+1}$.

In order to prove (1.5), we first note that, using (2.1), we have the equivalent formulation

$$(2.2) \quad N_n(k, m) = \sum_{u, v=0}^{\binom{n}{2}} c_n(u, v) p_n(k-u) p_n(m-v),$$

where $p_n(j)$ is the number of partitions of the integer j in n nonnegative integers. We now give a constructive proof that (2.2) holds. Note that we assume that $p_n(j) = 0$ if $j < 0$.

First suppose that $\pi = (i_1, i_2, \dots, i_n)$ has rise positions $j_1, j_2, \dots, j_r$; that is, $i_{j_k} < i_{j_k+1}$. Associate the matrix

$$A(\pi) = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}$$

where $j_0 = 0$ and $s_k = r$ ($j_0 + \dots + j_r + 1 \leq k \leq j_0 + \dots + j_r + j_{r+1}$) and t_k is the weight assigned to the element i_k when the permutation π is read. For example, if $\pi = (261435)$, then $A(\pi) = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 3 & 0 & 2 & 1 & 2 \end{pmatrix}$. If π has u weighted rises and v weighted readings, then it is immediate that $u = \sum s_i$ and $v = \sum t_i$. Thus, if $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$ are partitions of $k - u$ and $m - v$; i.e., if $\sum a_i = k - u$, $\sum b_i = m - v$, $0 \leq a_i \leq a_{i+1}$, and $0 \leq b_i \leq b_{i+1}$, then the matrix

$$A(\pi; \alpha, \beta) = A(\pi) + \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_{i_1} & b_{i_2} & \dots & b_{i_n} \end{pmatrix} = \begin{pmatrix} a_1 + s_1 & \dots & a_n + s_n \\ b_{i_1} + t_1 & \dots & b_{i_n} + t_n \end{pmatrix}$$

is a partition of the bipartite (k, m) . We claim that each partition counted by $N_n(k, m)$ is obtained in this way.

Since $a_i \leq a_{i+1}$ and $s_i \leq s_{i+1}$, it is immediate that the first row of $A(\pi; \alpha, \beta)$ is nondecreasing. Also these inequalities show that if $a_j + s_j = a_r + s_r$ with $1 \leq j < r \leq n$ then $a_j = a_r$ and $s_j = s_r$. This last entails that $i_j > i_r$ and thus that the element i_r is read on an earlier scan than the element i_j ; i.e., $t_j > t_r$. On the other hand, since $i_j > i_r$, we have that $b_{i_j} \geq b_{i_r}$ and so $b_{i_j} + t_j > b_{i_r} + t_r$, as desired.

In the next place, if $\pi = (i_1, \dots, i_n)$, $\pi' = (j_1, \dots, j_n)$ are permutations and α, α' and $\beta = (b_1, \dots, b_n)$, $\beta' = (c_1, \dots, c_n)$ are partitions such that $A(\pi; \alpha, \beta) = A(\pi'; \alpha', \beta')$, then we must show that $\pi = \pi'$, $\alpha = \alpha'$, and $\beta = \beta'$. We first note that the inequality $b_{i_r} + t_r \leq b_{i_{r+1}} + t_{r+1}$ is equivalent to $s_{r+1} = 1 + s_r$ or, what is the same thing, $i_r < i_{r+1}$. Using this remark, we see that π and π' have the same s vector and that this requires that $\alpha = \alpha'$. We now conclude the proof by induction on n . First choose r such that $i_r = 1$. Then $t_r = 0$ and $b_{i_r} + t_r = b_1 = c_{j_r} + t'_r \geq c_1$, where of course t'_r denotes the r th entry in the reading vector for π' . In the same way, we see that $c_1 \geq b_1$ and so $b_1 = c_1$. With this same choice of r we also note that $c_1 = b_1 = c_{j_r} + t'_r \geq c_1 + t'_r$ and so $t'_r = 0$. Thus the element j_r is read on the first scan of π' and if $j_r > 1$, there would be some integer $s < r$ for which $j_s = 1$. However, since $i_s > 1$, we have $c_1 = b_{i_s} + t_s > b_{i_s} \geq b_1 = c_1$, a contradiction. Thus $j_r = 1$ and by the inductive hypothesis we conclude that $\beta = \beta'$ and $\pi = \pi'$.

Finally, if

$$\Gamma = \begin{pmatrix} c_1 & \dots & c_n \\ d_1 & \dots & d_n \end{pmatrix}$$

is a partition of the bipartite (k, m) counted by $N_n(k, m)$, we must construct a permutation π and partitions α and β such that $A(\pi; \alpha, \beta) = \Gamma$. First order the second row of Γ as $d_{j_1} d_{j_2} \cdots d_{j_n}$ so that d_r precedes d_s if either $d_r < d_s$ or $d_r = d_s$ and $r < s$. Define the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \text{ by } \pi^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

and let

$$A(\pi) = \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}.$$

Then define the partitions $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ according to $a_r = c_r - s_r$ and $b_r = d_{j_r} - t_{j_r}$. It is immediate that $\Gamma = A(\pi; \alpha, \beta)$ and it is only necessary to verify that α and β defined in this way are partitions; i.e., that $0 \leq a_r \leq a_{r+1}$ and $0 \leq b_r \leq b_{r+1}$. These last inequalities follow rather easily. For example if r is the smallest index such that $a_{r+1} < 0$, then it follows that $c_r - s_r > c_{r+1} - s_{r+1}$ and from $c_r \leq c_{r+1}$ and $s_{r+1} \leq 1 + s_r$, we obtain $c_r = c_{r+1}$ and $s_{r+1} = 1 + s_r$. However, since $c_r = c_{r+1}$, we have $d_r > d_{r+1}$ so that $i_r > i_{r+1}$ and this requires that $s_r = s_{r+1}$, a contradiction. The proof that $a_r \leq a_{r+1}$ is identical and the proof that β is a partition is similar.

3. Computations. The combinatorial meaning assigned to the coefficients $d_n(u, v)$ by (1.6) makes it possible to compute their values for small values of n . The following tables indicate these values for $4 \leq n \leq 6$.

$u \backslash v$	0	1	2	3	4	5	6
0	1						
1		1	1	1			
2		1	2	1	1		
3		1	1	2	1	1	
4			1	1	2	1	
5				1	1	1	
6							1

$$n = 4$$

$u \backslash v$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1	1	1	1						
2		1	2	2	2	1	1				
3		1	2	3	3	3	2	1			
4		1	2	3	4	4	3	2	1		
5			1	3	4	6	4	3	1		
6			1	2	3	4	4	3	2	1	
7				1	2	3	3	3	2	1	
8					1	1	2	2	2	1	
9							1	1	1	1	
10											1

 $n = 5$

$u \backslash v$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		1	1	1	1	1										
2		1	2	2	3	2	2	1	1							
3		1	2	4	4	5	5	4	2	2						
4		1	3	4	7	7	8	7	6	3	2	1				
5		1	2	5	7	11	11	12	9	7	4	2				
6			2	5	8	11	15	14	13	10	7	3	2			
7			1	4	7	12	14	17	15	13	9	6	2	1		
8			1	2	6	9	13	15	17	14	12	7	4	1		
9				2	3	7	10	13	14	15	11	8	5	2		
10					2	4	7	9	12	11	11	7	5	2	1	
11					1	2	3	6	7	8	7	7	4	3	1	
12							2	2	4	5	5	4	4	2	1	
13								1	1	2	2	3	2	2	1	
14											1	1	1	1	1	
15																1

 $n = 6$

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