MAXIMUM ANTICHAINS: A SUFFICIENT CONDITION

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ABSTRACT. Given the finite partially ordered set $(Q, \leq)$, one might wish to know whether a maximal (nonextendible) antichain is a *maximum* antichain. Our result generalizes a theorem of Baker, which in turn constitutes a generalization of Sperner's lemma.

Introduction. Given a finite partially ordered set $(Q, \leq)$ and a maximal antichain A , one might seek conditions which force A to be a maximum antichain of Q . In this note we find some such conditions.

Sperner's lemma [3] states that a maximum antichain of the subset lattice of an n -element set consists of all the points at the $\lfloor n/2 \rfloor$ level. Our result implies both Sperner's lemma and a generalization thereof given by Baker [1]. Baker's result states that in a finite partially ordered set with a rank function, if the number of covering and covered elements is constant for elements of any given rank, then the elements at the most populous level constitute a maximum antichain.

Notation and Terminology. 1. $(Q, \leq)$ denotes a finite partially ordered set.

2. Let $a_1, a_2 \in Q$. If a_1 and a_2 are comparable, we write $a_1 R a_2$ (for " a_1 is related to a_2 "); otherwise we write $a_1 \nR a_2$.

3. A nonempty subset S of Q is a *chain* (*antichain*) iff, for any two distinct elements a_1, a_2 of S , $a_1 R a_2$ ($a_1 \nR a_2$).

4. S is a *maximal* chain (*antichain*) iff, for any $a \in Q - S$, $S \cup \{a\}$ is not a chain (*antichain*).

5. S is a *maximum* chain (*antichain*) iff, for any other chain (*antichain*) T , $|S| \geq |T|$.

6. For any nonempty subset S of Q , let $M(S)$ denote the collection of maximal chains of Q which intersect S .

Received by the editors January 18, 1973 and, in revised form, June 14, 1973.

AMS (MOS) subject classifications (1970). Primary 06A10; Secondary 06A20.

Key words and phrases. Maximum antichains, Sperner's lemma, partially ordered sets.

¹ Supported by the Fannie and John Hertz Foundation.

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Observe that if $S = \{q_1, \dots, q_n\}$ is an antichain, then $M(S)$ is the disjoint union of $M(q_j)$ for $j = 1, 2, \dots, n$.

Theorem. Let $\{q_1, \dots, q_n\}$ be a maximal antichain of Q . Assume that

(i) for each $q \in Q$ and $j \leq n \ni q R q_j$, $|M(q_j)| \leq |M(q)|$, and

(ii) $|M(Q)| \leq |\bigcup_{i=1}^n M(q_i)| + 1$.

Then $\{q_1, \dots, q_n\}$ is a maximum antichain.

Proof. Let $\{p_1, \dots, p_m\}$ be an antichain of Q . We show that $n \geq m$.

For $i \leq m$, let $Q_i = \{q_j : q_j R p_i\}$. $\{q_1, \dots, q_n\}$ is a maximal antichain so $Q_i \neq \emptyset$ for $i \leq m$. Assume $\{Q_1, \dots, Q_m\}$ does not have a system of distinct representatives (SDR). There is a smallest subcollection, say $\{Q_1, \dots, Q_{k+1}\}$, which does not have an SDR. (Here $1 \leq k < m$.) It follows easily from the theory of SDR's [2] that $|\bigcup_{i=1}^k Q_i| = k$ and that $Q_{k+1} \subseteq \bigcup_{i=1}^k Q_i$. To see this observe:

(a) $|\bigcup_{i \in F_0} Q_i| < |F_0|$ for some $F_0 \subseteq \{1, 2, \dots, k+1\}$ since $\{Q_1, \dots, Q_{k+1}\}$ does not have an SDR.

(b) $\{Q_i : i \in F\}$ has an SDR if $|F| \leq k$ and $F \subseteq \{1, 2, \dots, m\}$.

(c) From (b) it follows that $|\bigcup_{i \in F} Q_i| \geq |F|$ if $|F| \leq k$.

(d) Consequently, $F_0 = \{1, 2, \dots, k+1\}$ and $k \leq |\bigcup_{i=1}^k Q_i| \leq |\bigcup_{i=1}^{k+1} Q_i| < k+1$.

Relabelling if necessary, we may suppose $q_1, \dots, q_k$ is an SDR for $\{Q_1, \dots, Q_k\}$, where $q_i \in Q_i$ for $i \leq k$. Thus, for $i \leq k$, $p_i R q_i$ and so $|M(q_i)| \leq |M(p_i)|$.

All but at most one maximal chain which intersects $\{p_1, \dots, p_{k+1}\}$ intersects q_j for some $j \leq n$. Whenever there is such an intersection, $p_l R q_j$ for some $l \leq k+1$ which then implies that $q_j \in \bigcup_{i=1}^{k+1} Q_i$ and therefore that $j \leq k$. Consequently,

$$|M(\{p_1, \dots, p_{k+1}\})| - 1 \leq |M(\{q_1, \dots, q_k\})|.$$

$$\begin{aligned} |M(\{p_1, \dots, p_{k+1}\})| &= \left| \bigcup_{i=1}^{k+1} M(p_i) \right| = \sum_{i=1}^{k+1} |M(p_i)| \geq \sum_{i=1}^k |M(q_i)| + |M(p_{k+1})| \\ &= |M(\{q_1, \dots, q_k\})| + |M(p_{k+1})| \\ &\geq |M(\{p_1, \dots, p_{k+1}\})| - 1 + |M(p_{k+1})| \Rightarrow |M(p_{k+1})| \leq 1. \end{aligned}$$

There exists an $i \leq k \ni q_i R p_{k+1}$. Then $1 \leq |M(q_i)| \leq |M(p_{k+1})| \leq 1$. We also have $q_i R p_i$. Since there is only one maximal chain through q_j , $p_i R p_{k+1} \Rightarrow \Leftarrow$ contradiction. Hence $\{Q_1, \dots, Q_m\}$ has an SDR and so $n \geq m$.

An example constructed with the aid of Paul Klembeck shows that hypothesis (ii) cannot be deleted. A diagram of the example is as follows:

The points a_3 and a_4 constitute a maximal antichain satisfying (i). However, $\{a_1, a_2, a_3\}$ is the maximum antichain.

We indicate how this theorem implies the result of Baker. In Baker's theorem, each point $q \in Q$ at level l lies below b_l points at level $l+1$, while in turn covering a_l points at level $l-1$. A direct argument counting the number of maximal chains passing through a given point at, say, level l shows that any two points at level l have the same number of maximal chains passing through them. Also, note that every maximal chain of Q passes through some point of level l . Therefore, the number of maximal chains passing through a given point at level l is M/p_l , where M is the total number of maximal chains of Q and p_l is the number of points of Q at level l . If we therefore let $\{q_1, \dots, q_n\}$ be the collection of points of Q at the most populous level, this set will be a maximal antichain of Q satisfying conditions (i) and (ii).

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