THE RADON-NIKODYM PROPERTY AND DENTABLE SETS IN BANACH SPACES

W. J. DAVIS AND R. R. PHELPS¹

ABSTRACT. In order to prove a Radon-Nikodym theorem for the Bochner integral, Rieffel [5] introduced the class of "dentable" subsets of Banach spaces. Maynard [3] later introduced the strictly larger class of "*s*-dentable" sets, and extended Rieffel's result to show that a Banach space has the Radon-Nikodym property if and only if every bounded nonempty subset of E is *s*-dentable. He left open, however, the question as to whether, in a space with the Radon-Nikodym property, every bounded nonempty set is dentable. In the present note we give an elementary construction which shows this question has an affirmative answer.

Definitions. A Banach space E has the *Radon-Nikodym property* if for each totally finite positive measure space (X, Σ, μ) and each E -valued, μ -continuous measure m on Σ with $|m|(X) < \infty$, there is a Bochner integrable f from X to E such that $m(E) = \int_E f d\mu$ ($E \in \Sigma$).

A subset A of E is *dentable* if for every $\epsilon > 0$, there exists $x \in A$ such that $x \notin \text{cl co}(A \setminus S_\epsilon(x))$. [Here $\text{co} B$ denotes the convex hull of B , $\text{cl co} B$ is its closure and $S_\epsilon(x) = \{y \in E: \|x - y\| \leq \epsilon\}$.] A bounded set A is called *s-dentable* if for each $\epsilon > 0$ there exists $x \in A$ such that $x \notin s(A \setminus S_\epsilon(x))$. [Here $s(B) = \{\sum_{i=1}^\infty \lambda_i x_i: \lambda_i \geq 0, \sum \lambda_i = 1, \{x_i\} \subset B\}$.] A point $x \in A$ is a *denting* [*s-denting*] point if for all $\epsilon > 0$, $x \notin \text{cl co}(A \setminus S_\epsilon(x))$ [$x \notin s(A \setminus S_\epsilon(x))$].

Dentable sets are *s*-dentable, and Maynard has given an example of a bounded set which is *s*-dentable but not dentable. Rieffel has shown that if A is not dentable, then neither is $\text{cl co } A$. The analogous assertion fails for *s*-dentability: The closed unit ball of $C([0, 1])$ is not dentable [5], but the constant-1 function is an *s*-denting point. By Lemma 2 below, its interior is not *s*-dentable.

Lemma 1. *A subset A of E is not dentable if and only if there exists*

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$\epsilon > 0$ such that $A \subset \text{cl co}(A \setminus S_\epsilon(x))$ for each $x \in A$. If A is closed and convex, this is equivalent to $A = \text{cl co}(A \setminus S_\epsilon(x))$ for each $x \in A$.

Proof. One implication is trivial. For the other, suppose A is not dentable. Then there exists $2\epsilon > 0$ such that for each $y \in A$, $y \in \text{cl co}(A \setminus S_{2\epsilon}(y))$. Suppose that $x, y \in A$ and $\|x - y\| > \epsilon$. Then $y \in \text{cl co}(A \setminus S_\epsilon(x))$. On the other hand, if $\|x - y\| \leq \epsilon$, then $S_\epsilon(x) \subset S_{2\epsilon}(y)$ so that $y \in \text{cl co}(A \setminus S_{2\epsilon}(y)) \subset \text{cl co}(A \setminus S_\epsilon(x))$, completing the proof.

Lemma 2. *Suppose C is a closed convex set in E with nonempty interior (denoted by $\text{int } C$) and suppose C is not dentable. Then there exists $\epsilon > 0$ such that for each $x \in C$, $\text{int } C \subset \text{co}[\text{int } C \setminus S_\epsilon(x)]$. In particular, $\text{int } C$ is not s -dentable.*

Proof. By Lemma 1, there exists $\epsilon > 0$ such that $C = \text{cl co}(C \setminus S_\epsilon(x))$ for each $x \in C$. Let $J_x = C \setminus S_\epsilon(x)$ so that $\text{int } J_x = \text{int } C \setminus S_\epsilon(x)$. For ϵ sufficiently small, $\text{int } J_x \neq \emptyset$ for each $x \in C$. Fix x and let $J = J_x$. Then $C = \text{cl co } J$ and we want to show that $\text{int}(\text{cl co } J) \subset \text{co}(\text{int } J)$. Note that $J \subset \text{cl}(\text{int } J)$: If $y \in J$, then $y \in C$ and $\|y - x\| > \epsilon$. Let $z \in \text{int } C$, so that $[z, x) \subset \text{int } C$ and there exists $u \in [z, x) \cap S_\epsilon(x)$. Therefore, $[u, y) \subset \text{int } C$ so for some $v \in [u, y)$ we have $[v, y) \subset \text{int } C \setminus S_\epsilon(x)$. Thus $y \in \text{cl}(\text{int } J)$. It now follows that $\text{co } J \subset \text{cl co}(\text{int } J)$. Taking the interior of each side, we conclude that

$$\text{int}(\text{cl co } J) = \text{int}(\text{co } J) \subset \text{int}(\text{cl co}(\text{int } J)) = \text{co}(\text{int } J).$$

The equalities follow from the fact that the interior of a convex set coincides with the interior of its closure.

Proposition. *If E contains a bounded nonempty set which is not dentable, then it contains a bounded closed convex and symmetric set C which is not dentable and which has nonempty interior. In particular, E can be renormed so that the new unit ball is not dentable and the interior of the new unit ball is not s -dentable.*

Proof. If A is a bounded nonempty set which is not dentable, then the same is true of the sets $A_1 = A \cup (-A)$ (definition), $A_2 = \text{cl co } A_1$ (Rieffel [5, Proposition 2]) and $A_3 = S + A_2$, where S is the closed unit ball of E (easy computation). Let C be the closure of A_3 . Again, by Rieffel's proposition, C is not dentable. By Lemma 2, $\text{int } C$ is not s -dentable.

What we have shown is that every bounded subset of E is dentable if and only if every bounded subset of E is s -dentable. This yields the following corollary.

Corollary: *A Banach space E has the Radon-Nikodym property if and only if every bounded nonempty set in E is dentable.*

J. Diestel has raised the question of the relationship between the Radon-Nikodym property and the Krein-Milman property [every closed bounded convex set is the closed convex hull of its extreme points]. It is known that a space has both properties if it is reflexive or is a separable conjugate space [1], [4]. The Proposition shows that every bounded nonempty subset of E is dentable if and only if the unit ball of every Banach space isomorphic to E is dentable. Is the analogous assertion true for the Krein-Milman property? That is, if E contains a closed, bounded nonempty convex set which is not the closed convex hull of its extreme points, can E be renormed so that its new unit ball is not the closed convex hull of its extreme points? The answer is affirmative for the spaces (m) and $L_1([0, 1])$ [2]. In spaces with the Radon-Nikodym property does every closed bounded convex set have a denting point (s -denting point, extreme point)?

Added in proof. Since this paper was submitted, a number of related results have appeared. Huff [6] has given an independent proof of the above corollary, by modifying the proof of Maynard's main theorem [3]. Lindenstrauss has used the corollary to show (cf. [7]) that the Radon-Nikodym property implies the Krein-Milman property, and the question concerning denting points has also received an affirmative answer in [7]. Huff and Morris [8] have shown that the Krein-Milman property implies the Radon-Nikodym property in conjugate spaces, but this implication remains open for arbitrary spaces. Finally, Edgar [9] has generalized Lindenstrauss' result by obtaining a Choquet-type representation theorem for bounded closed convex subsets of a Banach space with the Radon-Nikodym property.

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO
43210

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASH-
INGTON 98195