PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 1, July 1974

PERTURBATIONS OF SEMI-FREDHOLM OPERATORS BY OPERATORS CONVERGING TO ZERO COMPACTLY

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ABSTRACT. Let $\{K_n\}$ be a sequence of bounded linear operators mapping a Banach space X into a Banach space such that $K_n \to 0$ strongly and $\{K_n x_n\}$ is relatively compact for every bounded sequence $\{x_n\} \subset X$; e.g., $\|K_n\| \to 0$. Given T a semi-Fredholm operator, it is shown that for all sufficiently large n, $T + K_n$ has nullity and deficiency not exceeding that of T while the index of $T + K_n$ equals that of T. Properties of the minimum modulus of $T + K_n$ are also given.

Let X and Y be Banach spaces. A sequence $\{K_n\}$ of bounded linear operators mapping X into Y is said to converge to zero compactly, written $K_n \xrightarrow{c} 0$, if

(1) $K_n x \to 0$ for all $x \in X$;

(2) $\{K_n x_n\}$ is relatively compact for every bounded sequence $\{x_n\} \subset X$.

Clearly, $\|K_n\| \to 0$ implies $K_n \xrightarrow{c} 0$. If $\bigcup_n K_n S$ is relatively compact, where S is the 1-ball in X, then (2) is satisfied. In this case $\{K_n\}$ is called collectively compact and was intensively studied in [1].

In this paper we present properties of the nullity, deficiency and index of operators of the form $T + K_n$, where T is a semi-Fredholm operator and $K_n \xrightarrow{c} 0$. We obtain theorems analogous to those given in [3] and [4], where the requirement was that K_n be "small enough" in norm. We generalize the results and simplify the proofs appearing in [5] where X and Y were assumed to be Hilbert spaces with T a bounded semi-Fredholm operator.

Throughout this paper T is assumed to be a closed linear operator with domain $\mathfrak{D}(T) \subset X$ and range $\mathfrak{R}(T)$ a closed *subspace of Y.*

$\alpha(T) = \dim \mathfrak{N}(T)$, where $\mathfrak{N}(T)$ is the kernel of T and $\beta(T) = \operatorname{co\,dim} \mathfrak{R}(T)$. If, in addition, $\alpha(T)$ or $\beta(T)$ is finite, T is called a semi-

Received by the editors May 21, 1973.

AMS (MOS) subject classifications (1970). Primary 47A55, 47B30.

Key words and phrases. Semi-Fredholm, index, nullity, deficiency, minimum modulus.

Fredholm operator with index $\kappa(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, T is called a Fredholm operator.

For properties of semi-Fredholm operators the reader is referred to [3] and [4].

Preliminary remarks.

I. $\mathcal{R}(T)$ is closed if and only if $\gamma(T) = \inf \|Tx\|/d(x, \mathfrak{N}(T)) = \|\hat{T}^{-1}\|^{-1} > 0$, where $d(x, \mathfrak{N}(T))$ is the distance from x to $\mathfrak{N}(T)$ and $\hat{T}$ is the 1-1 operator induced by T.

II. If $\{A_n\}$, a sequence of bounded linear operators on X with range in Y, converges strongly to zero, then $\{\|A_n\|\}$ is bounded and $\{A_n\}$ converges to zero uniformly on totally bounded sets.

III. If $\{y_n\}$ is a bounded sequence in $\mathcal{R}(T)$, then there exists a bounded sequence $\{x_n\}$ such that $Tx_n = y_n$.

This follows readily from $y_n = Tv_n$ and $\|y_n\| = \|Tv_n\| \geq \gamma(T)d(v_n, \mathfrak{N}(T))$.

IV. If $\alpha(T) < \infty$ and $\{x_n\}$ is a bounded sequence such that $\{Tx_n\}$ converges, then $\{x_n\}$ has a convergent subsequence; for, since $\mathcal{R}(T)$ is closed, $Tx_n \to Tx$ and therefore $x_n + \mathfrak{N}(T) = \hat{T}^{-1}Tx_n \to \hat{T}^{-1}Tx = x + \mathfrak{N}(T)$ in $X/\mathfrak{N}(T)$. Thus there exists $z_n \in \mathfrak{N}(T)$ such that $x_n + z_n \to x$. Since $\{z_n\}$ is bounded in finite dimensional space $\mathfrak{N}(T)$, it, and therefore $\{x_n\}$, has a convergent subsequence.

Basic Lemma [2, p. 190]. *If M and N are subspaces of X and* $\dim M > \dim N$, *there exists an $m \in M$ such that $1 = \|m\| = d(m, N)$.*

Throughout the remainder of this paper, $K_n \xrightarrow{c} 0$.

Lemma 1. *If $\alpha(T) < \infty$ and $\mathfrak{N}(T)$ is complemented in X by a closed subspace M, then there exists a p and $c > 0$ such that for $n \geq p$, $T_M + K_n$ is 1-1 and $\gamma(T_M + K_n) \geq c$, where T_M is the restriction of T to $M \cap \mathfrak{D}(T)$.*

Proof. Suppose $\{\gamma(T_M + K_n)\}$ has a subsequence converging to zero. For simplicity, let $\gamma(T_M + K_n) \to 0$. There exists $\{m_n\} \subset M$ such that $\|m_n\| = 1$ and $(T + K_n)\, m_n \to 0$. Since $K_n \xrightarrow{c} 0$, $\{K_n m_n\}$, and therefore $\{Tm_n\}$, has a convergent subsequence. Thus by IV, $\{m_n\}$ has a convergent subsequence and by II, $\{K_n m_n\}$, and therefore $\{Tm_n\}$, has a subsequence converging to zero. This is impossible since T_M has a bounded inverse. This argument also shows that $T_M + K_n$ is 1-1 for sufficiently large n; otherwise, a sequence $\{m_n\}$ with the above properties would obviously exist which leads to a contradiction.

The assumption $K_n \xrightarrow{c} 0$ does not even imply that the sequence $\{K_n'\}$ of conjugate operators converges strongly. The following simple example taken from [1] confirms this.

Take $X = Y = l_2$. Define $K_n x = x_n e_1$, where $x = \Sigma_1^\infty x_i e_i$, $\{e_i\}$ the usual set of unit vectors. Then $K_n' e_1 = e_n$ does not converge in l_2.

In light of this example, it is somewhat surprising that the following "dual" lemma holds.

Lemma 2. *Let $\mathfrak{D}(T)$ be dense in X. If $\mathfrak{R}(T)$ is complemented in Y by a closed subspace W, then there exists a p and $c > 0$ such that for $n \geq p$, $T' + K_n'$ is $1 - 1$ on $W^\circ = \{y' \in Y' : y'W = 0\}$ and $\gamma(T_0' + K_n') \geq c$, where T_0' is the restriction of T' to $W^\circ \cap \mathfrak{D}(T')$.*

Proof. $Y = \mathfrak{R}(T) \oplus W$ and $Y' = \mathfrak{R}(T)^\circ \oplus W^\circ$. Suppose $\gamma(T_0' + K_n')$ has a subsequence converging to zero. For simplicity, let $\gamma(T_0' + K_n') \to 0$. There exists $\{y_n'\} \subset W^\circ$ such that $1 = \|y_n'\|$ and $(T' + K_n')y_n' \to 0$. Choose y_n so that $1 = \|y_n\|$ and $y_n' y_n \geq ½$. Now $y_n = Tv_n + w_n$, $w_n \in W$ and $\{Tv_n\}$ is bounded since $\mathfrak{R}(T)$ is closed and complemented by W. Hence by III, there exists a bounded sequence $\{x_n\}$ such that $Tx_n = Tv_n$. Furthermore, $y_n' v \to 0$ for all $v \in Y$. To see this, $y_n' Tx = (T' + K_n')y_n'(x) - y_n' K_n x \to 0$. Since y_n' is in W° and $\mathfrak{R}(T)$ is complemented by W, $y_n' v \to 0$ for all $v \in Y$. Now

$$(*) \qquad ½ \leq y_n' y_n = (T' + K_n')y_n'(x_n) - y_n' K_n x_n.$$

Since $\{x_n\}$ is bounded, $\{K_n x_n\}$ is totally bounded which, together with the observation that $y_n' v \to 0$ for all $v \in Y$, implies that $\{y_n' K_n x_n\}$ converges to zero. Therefore $(*)$ cannot hold since $(T' + K_n')y_n' \to 0$.

The above argument also shows that $T' + K_n'$ is $1 - 1$ on W° for all sufficiently large n; otherwise a sequence $\{y_n'\}$ with the above properties would obviously exist which leads to a contradiction.

Theorem 1. *Suppose $\alpha(T) < \infty$. There exists a p such that:*

(1) *$T + K_n$ has a closed range and $\alpha(T + K_n) \leq \alpha(T)$, $n \geq p$.*

(2) *$\alpha(T + K_n) = \alpha(T)$, $n \geq p$, if and only if $\inf_{n \geq p} \gamma(T + K_n) > 0$. In this case, $X = M \oplus \mathfrak{N}(T + K_n)$, $n \geq p$, where $\mathfrak{N}(T)$ is complemented by the closed subspace M.*

Proof. $X = M \oplus \mathfrak{N}(T)$ for some closed subspace M. Let p and $c > 0$ be as in Lemma 1 and $n \geq p$. Then $(T + K_n)M$ is closed by I and the finite dimensionality of $\mathfrak{N}(T)$ implies $\mathfrak{R}(T + K_n) = (T + K_n)M + K_n\mathfrak{N}(T)$ is closed. Moreover, by Lemma 1, $M \cap \mathfrak{N}(T + K_n) = \{0\}$. Hence $X = M \oplus \mathfrak{N}(T) \supset M \oplus \mathfrak{N}(T + K_n)$ which implies $\alpha(T + K_n) \leq \alpha(T)$.

Suppose $\alpha(T + K_n) = \alpha(T)$, $n \geq p$. Then $X = M \oplus \mathfrak{N}(T + K_n)$ and for $x = m_n + z_n$, $m_n \in M$, $z_n \in \mathfrak{N}(T + K_n)$, we have by Lemma 1 that

$$\|(T + K_n)x\| = \|(T + K_n)m_n\| \geq c\,\|m_n\|$$
$$\geq cd(m_n, \mathfrak{N}(T + K_n)) = cd(x, \mathfrak{N}(T + K_n)).$$

Thus $\gamma(T + K_n) \geq c > 0$, $n \geq p$. Conversely, suppose $\gamma(T + K_n) \geq c > 0$, $n \geq p$, but that $\alpha(T + K_n) \neq \alpha(T)$. Then, from (1), $\alpha(T + K_n) < \alpha(T)$. By the basic lemma there exists $\{z_n\} \subset \mathfrak{N}(T)$ such that $1 = \|z_n\| = d(z_n, \mathfrak{N}(T + K_n))$. Hence for $n \geq p$,

$$(*) \qquad 0 < c = cd(z_n, \mathfrak{N}(T + K_n)) \leq \|(T + K_n)z_n\| = \|K_n z_n\|.$$

Since $\mathfrak{N}(T)$ is finite dimensional, $\{z_n\}$ has a convergent subsequence and therefore by II, $\{K_n z_n\}$ has a subsequence converging to zero, contradicting $(*)$.

Theorem 2. *If $\mathfrak{R}(T)$ is complemented in Y (by a closed subspace) and T is densely defined, there exists a p such that for $n \geq p$, $\alpha(T' + K_n') \leq \alpha(T')$.*

If $\beta(T) < \infty$, there exists a p such that:

(i) *$T + K_n$ has closed range with $\beta(T + K_n) \leq \beta(T)$, $n \geq p$.*

(ii) *$\beta(T + K_n) = \beta(T)$, $n \geq p$, implies $\inf_{n \geq p} \gamma(T + K_n) > 0$.*

Proof. Let W and p be chosen as in Lemma 2 with $n \geq p$. Then $Y' = \mathfrak{R}(T)^\circ \oplus W^\circ = \mathfrak{N}(T') \oplus W^\circ$. Since $T' + K_n'$ is 1-1 on W°, $Y' \supset \mathfrak{N}(T' + K_n') \oplus W^\circ$. Thus $\alpha(T' + K_n') \leq \alpha(T')$.

(i) By replacing X by $\mathrm{cl}(\mathfrak{D}(T))$, if necessary, we may assume T is densely defined. Since $\beta(T) < \infty$, there exists a p and W as in Lemma 2. For $n \geq p$, $(T' + K_n')W^\circ$ is closed by preliminary remark I. Since $\alpha(T') = \beta(T) < \infty$, $(T' + K_n')Y' = (T' + K_n')W^\circ + K_n'\mathfrak{N}(T')$ is closed; i.e., $T' + K_n'$ has a closed range and therefore $T + K_n$ has a closed range. Thus by what we have already shown,

$$\beta(T + K_n) = \alpha(T' + K_n') \leq \alpha(T') = \beta(T), \qquad n \geq p.$$

(ii) Suppose $\beta(T + K_n) = \beta(T) < \infty$ or equivalently $\alpha(T' + K_n') = \alpha(T')$, $n \geq p$, with p and c chosen as in Lemma 2. Then $Y' = \mathfrak{N}(T') \oplus W^\circ = \mathfrak{N}(T' + K_n') \oplus W^\circ$. Thus for $y' = z_n' + w_n'$, $z_n' \in \mathfrak{N}(T' + K_n')$, $w_n' \in W^\circ$, we have

$$\|(T' + K_n')y'\| = \|(T' + K_n')w_n'\| \geq c\,\|w_n'\| \geq c\,d(y', \mathfrak{N}(T' + K_n')).$$

Hence $\gamma(T + K_n) = \gamma(T' + K_n') \geq c$, $n \geq p$.

Theorem 3. *Let T be a Fredholm operator. Then $\gamma(T + K_n)$ is bounded away from zero for all sufficiently large n if and only if $\alpha(T + K_n) = \alpha(T)$ and $\beta(T + K_n) = \beta(T)$ for all sufficiently large n.*

Proof. By replacing X by $\mathrm{cl}(\mathfrak{D}(T))$, if necessary, we may assume T is densely defined. $Y = \mathfrak{R}(T) \oplus W$, W finite dimensional. Suppose $\gamma(T + K_n) \geq c > 0$ for all but a finite number of n but that $\beta(T + K_n) \neq \beta(T)$ for infinitely many n. Then by (i) of Theorem 2, $\beta(T + K_n) < \beta(T)$ for infinitely many n. For simplicity, suppose $\beta(T + K_n) < \beta(T)$ and $\gamma(T + K_n) \geq c$ for $n \geq p$, where p is chosen so that Lemma 2 holds. Thus there exists $y_n \in \mathfrak{R}(T + K_n) \cap W$, $\|y_n\| = 1$. Since $\|y_n\|$ is bounded and $\gamma(T + K_n) \geq c > 0$ it follows that there exists a bounded sequence $\{x_n\}$ such that $y_n = (T + K_n)x_n$. Now $\{y_n\}$ has a convergent subsequence since W is finite dimensional; say $y_{n'} \to y \in W$. Since $\{K_{n'}x_{n'}\}$ has a convergent subsequence, so does $\{Tx_{n'}\}$. Thus by preliminary remarks IV and II, $\{x_{n'}\}$ has a convergent subsequence and $\{K_{n'}x_{n'}\}$ has subsequence $\{K_{n''}x_{n''}\}$ converging to zero. Hence $y = \lim y_{n''} = \lim Tx_{n''} \in \mathfrak{R}(T)$, which shows that y is in $\mathfrak{R}(T) \cap W = (0)$. This is impossible since $\|y\| = 1$. The rest of the theorem follows from Theorem 1.

Theorem 4. *Let T be a semi-Fredholm operator. There exists a p such that for $n \geq p$,*

(1) *$T + K_n$ is semi-Fredholm,*

(2) $\alpha(T + K_n) \leq \alpha(T)$,

(3) $\beta(T + K_n) \leq \beta(T)$,

(4) $\kappa(T + K_n) = \kappa(T)$.

Proof. The first three conclusions are contained in Theorems 1 and 2. There exists a p such that for all $\lambda \in [0, 1]$ and $n \geq p$, $T + \lambda K_n$ is semi-Fredholm. If this is not the case, there exists a subsequence $\{K_{n'}\}$ and a sequence $\lambda_n \in [0, 1]$ such that $T + \lambda_n K_{n'}$ is not semi-Fredholm. This is impossible by Theorems 1 and 2 since $\lambda_n K_{n'} \xrightarrow{c} 0$. Given $n \geq p$, define ϕ on $[0, 1]$ with values in the set of extended integers with the discrete topology by $\phi(\lambda) = \kappa(T + \lambda K_n)$. By [3, V.1.6], ϕ is continuous, and since

$[0, 1]$ is connected, ϕ is constant. In particular $\kappa(T) = \phi(0) = \phi(1) = \kappa(T + K_n)$.

Remark. In Bull. Austral. Math. Soc. 8 (1973), 279–287, Lo proved the following theorem.

Let T, T_n be bounded linear operators on X, T compact and $\|T_n - T\| \to 0$. Let $\mu \neq 0$ be an eigenvalue of T, and let μ_n be eigenvalues of T_n such that $\mu_n \to \mu$. Then the following are equivalent:

(a) $\dim \mathfrak{N}(\mu_n - T_n) = \dim \mathfrak{N}(\mu - T)$ *eventually*;

(b) *for every x in $\mathfrak{N}(\mu - T)$, $\|x\| = 1$, there is a sequence $\{x_n\}$ such that $x_n \in \mathfrak{N}(\mu_n - T_n)$ and $x_n \to x$.*

The above result is a very special case of Theorem 1. For consider $\mu_n - T_n = \mu - T + K_n$ where $K_n = \mu_n - \mu + T - T_n$. Then $\|K_n\| \to 0$ and therefore $K_n \xrightarrow{c} 0$. Since T is compact, $\mu - T$ is a Fredholm operator. Assuming (a), we have, by Theorem 1, the existence of a $c > 0$ such that for all n sufficiently large and $x \in \mathfrak{N}(\mu - T)$,

$$\|K_n x\| = \|(\mu_n - T_n)x\| \geq c d(x, \mathfrak{N}(\mu_n - T_n)).$$

Since $K_n x \to 0$, (b) follows. On the other hand, if (b) holds, then Theorem 1 together with the basic lemma imply (a).

The proof shows that T need not be compact but only that $\mu - T$ be semi-Fredholm with $\alpha(\mu - T) < \infty$. Moreover, $\|T_n - T\| \to 0$ can be replaced by $T_n - T \xrightarrow{c} 0$. If we stipulate that $\mu - T$ be Fredholm, then to (a) in the above theorem we may add $\beta(\mu_n - T_n) = \beta(\mu - T)$ for all n sufficiently large. This is a consequence of Theorem 3.

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