COMPACT AND QUASINORMAL COMPOSITION OPERATORS

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ABSTRACT. Let C_ϕ be a composition operator on $L^2(\lambda)$, where λ is a σ -finite measure on a set X . If X is nonatomic, then Ridge proved that no one-to-one composition operator C_ϕ with dense range is compact. This result is generalized in the paper by removing one-to-one and dense range conditions. The quasinormal composition operators are also characterized in terms of commutativity with the multiplication operator induced by the Radon-Nikodym derivative of the measure $\lambda\phi^{-1}$ with respect to λ .

1. Introduction. Some results concerning compact and quasinormal composition operators on $L^2(\lambda)$ are reported in this paper, where λ is a σ -finite measure on some set X . Ridge [1] proved that if X is nonatomic, then no one-to-one composition operator with dense range is compact. We have generalized this result in §2 by removing one-to-one and dense range conditions. In §3, we have proved a theorem on quasinormal composition operators.

A composition operator C_ϕ on $L^2(\lambda)$ induced by a measurable transformation of ϕ on X into itself is defined by $C_\phi f = f \circ \phi$.

2. Compact operators.

Theorem 1. *Let X be a nonatomic σ -finite measure space. Then no composition operator on $L^2(\lambda)$ is compact.*

First we shall prove the following lemma.

Lemma. *Let X be a σ -finite, nonatomic measure space. Let E be a nonnull measurable subset of X . Let $\{E_n\}$ be a sequence of nonnull measurable sets such that $E_{n+1} \subset E_n \subset E$ for all n , and $\lambda(E_n) < 1/n$. Let $g \in L^2(E)$. Then*

$$(\lambda(E_n))^{-1/2} \int_{E_n} g d\lambda \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

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Proof. Suppose there exists a $g \in L^2(E)$ such that $(\lambda(E_n))^{-1/2} \int_{E_n} g d\lambda$ does not tend to 0 as n tends to ∞ . We can assume that $\|g\| = 1$ and g is real valued. Then there exists an $\epsilon > 0$ such that

$$\limsup_{n \rightarrow \infty} \left| (\lambda(E_n))^{-1/2} \int_{E_n} g d\lambda \right| = 2\epsilon > 0.$$

Let e_n denote the function $(\lambda(E_n))^{-1/2} \chi_{E_n}$. Let $\mathcal{F} = \{n: n \in N \text{ and } |\langle g, e_n \rangle| > \epsilon\}$. Let $F \subset N$. Then by Bessel's inequality we have

$$(*) \quad \sum_{n \in F} |\langle g, e_n \rangle|^2 \leq 1 + \sum_{m, n \in F; m \neq n} \langle g, e_n \rangle \langle g, e_m \rangle \langle e_n, e_m \rangle.$$

Now choose a positive integer K such that $\epsilon^2 > 1/K$. Select a subset F of $\mathcal{F}$ such that cardinality of F is $2K$ and $\langle e_n, e_m \rangle < (1/2)(1/2K(2K-1))$ for all $m, n \in F$, $m \neq n$. From this it follows that $\sum_{n \in F} |\langle g, e_n \rangle|^2 \geq 2$ and $\sum_{n, m \in F; n \neq m} \langle e_n, e_m \rangle \leq 1/2$; hence from (*) we obtain $2 \leq 1 + 1/2 = 3/2$, which is a contradiction. Thus proof of the Lemma is complete.

Proof of the theorem. Let C_ϕ be a composition operator on $L^2(\lambda)$. Let f_0 denote the Radon-Nikodym derivative of the measure $\lambda\phi^{-1}$ with respect to λ . Since λ is nonatomic, we can choose a nonnull subset E such that f_0 is bounded away from zero on E . Also, we can choose a sequence $\{E_n\}$ of E such that $E_{n+1} \subset E_n$ and $0 < \lambda(E_n) < 1/n$ for all n . Consider the sequence $\{e_n\}$ of functions defined by $e_n = (\lambda(E_n))^{-1/2} \chi_{E_n}$. By the previous lemma $e_n \rightarrow 0$ weakly. It can be shown that $\{C_\phi e_n\}$ does not tend to 0 strongly. Hence C_ϕ is not compact. This completes the proof of the theorem.

Corollary 1. Let λ be a regular continuous Borel measure on R . Then no composition operator on $L^2(\lambda)$ is compact.

The following theorem also generalizes a result of Ridge [1].

Theorem 2. Let X be any σ -finite measure space, and let C_ϕ be a bounded operator on $L^2(\lambda)$. Then C_ϕ is compact implies that X is an atomic measure space.

Proof. We can write $X = X_1 \cup X_2$ where X_1 does not contain any atoms, X_2 is a countable union of atoms and X_1 and X_2 are disjoint [3]. Let λ_2 be the restriction of λ to X_2 , and let $\lambda_1 = \lambda - \lambda_2$. Since $\lambda\phi^{-1}$ is absolutely continuous with respect to λ , we can conclude that $X_2 \subset \phi^{-1}(X_2)$ a.e. If we write $L^2(\lambda) = L^2(\lambda_1) \oplus L^2(\lambda_2)$, then it follows from the above inclusion

that $C_\phi L^2(\lambda_1) \subset L^2(\lambda_1)$. Since C_ϕ is compact, from the previous theorem it follows that $\lambda_1 = 0$. Hence $\lambda = \lambda_2$. Hence X is an atomic measure space (since λ_2 is an atomic measure). The proof is complete.

3. Quasinormal composition operators.

Theorem 3. A composition operator C_ϕ on $L^2(\lambda)$ is quasinormal if and only if it commutes with M_{f_0} , where M_{f_0} is the multiplication operator induced by f_0 ($= d\lambda\phi^{-1}/d\lambda$).

Proof. First we will prove that $C_\phi^* C_\phi = M_{f_0}$. For any f and g in $L^2(\lambda)$

$$\langle C_\phi^* C_\phi f, g \rangle = \langle C_\phi f, C_\phi g \rangle = \int f \cdot \bar{g} d\lambda\phi^{-1} = \int f_0 f \cdot \bar{g} d\lambda = \langle M_{f_0} f, g \rangle.$$

Hence $C_\phi^* C_\phi = M_{f_0}$.

Now suppose C_ϕ is quasinormal. Then $C_\phi^* C_\phi C_\phi = C_\phi C_\phi^* C_\phi$, from which we get $M_{f_0} C_\phi = C_\phi M_{f_0}$. Conversely, if $C_\phi M_{f_0} = M_{f_0} C_\phi$, then, since $M_{f_0} = C_\phi^* C_\phi$, we get $C_\phi C_\phi^* C_\phi = C_\phi^* C_\phi C_\phi$, which shows that C_ϕ is quasinormal.

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