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ON THE DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT. Let $f(z)$ be any transcendental entire function. Let r_k denote the absolute value of the zero z_k of $f^{(k)}(z)$ which is nearest to the origin. Ålander, Erdős and Rényi, and Pólya have investigated the relation between r_k and the growth of the function $f(z)$. Let s_k denote the largest disk about the origin where $f^{(k)}(z)$ is univalent. Boas, Levinson, and Pólya have obtained some relations between the growth of the function $f(z)$ and s_k. Recently Shah and Trimble have sharpened the results of Boas and Pólya. We present here results in a different direction, generalizing the above quoted results. We also present results connecting the zero-free disks and the univalent disks about the origin of the normalized remainders of $f(z)$ with the growth of $f(z)$.

1. Introduction. There is extensive literature on the existence of zero-free disks for a sequence of derivatives of an entire function, as well as on disks of univalence for derivatives. Recently it has become clear that there are analogous results for a class of operators much more general than the operator of differentiation, the so-called D-operators (cf. [6]). These are defined as follows.

Let $\{d_p\}_{p=1}^{\infty}$ denote a nondecreasing sequence of positive numbers and let the operator D transform the function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ into $Df(z) = \sum_{j=0}^{\infty} d_{j+1} a_{j+1} z^j$. In general for $k = 0, 1, 2, \cdots$,

$$D^k f(z)^{1} = \sum_{j=0}^{\infty} \frac{e_j}{e_{k+1}} a_{k+j} z^j, \quad \text{where } e_0 = 1 \text{ and } e_j = (d_1 d_2 \cdots d_j)^{-1}.$$

For $d_n = n$, D is the ordinary derivative; for $d_n = 1$, D is the shift operator σ, whose iterates are the normalized remainders of the power series of the function; much less has been known about zero-free disks for $\sigma^k f(z)$ than for

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[1] We restrict that $\limsup_{n\to\infty} |a_n/e_n|^{1/n} < \infty$; this restriction assures that each of $f, Df, \cdots, D^k f$, is entire (cf. [6, p. 350]).

$f^{(k)}(z)$ (cf. [7]). By studying D-operators in general we not only show that many known results for derivatives appear as special cases of a general theory, but we also obtain the unexpected result that for the D-operators corresponding to certain sequences $\{d_p\}$ the functions $D^k f(z)$ have, so to speak, much scarcer zeros than is the case for $f^{(k)}(z)$. For example, if the lower order of f is less than α and $d_k = k^{1/\alpha}$ $(\alpha > 0)$, then Theorem 2 of this paper asserts that anywhere in the plane for every disk of arbitrarily large radius there exist infinitely many k where $D^k f(z) \neq 0$.

Throughout our work we restrict d_j by

(1) $$d_{j+k} \leq d_{k+1} d_j \quad \text{where } k, j = 1, 2, 3, 4, \cdots.$$

2. Lemmas.

Lemma 1 [2, p. 82]. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be any entire function. Then for an infinity of k and $j = 0, 1, 2, 3, \cdots$, we have $|a_{k+j}| \leq |a_k| |a_k|^{j/k}$.*

Lemma 2 [9, (39)]. *Let $f(z)$ be an arbitrary entire function, $M(r) = \max_{|z|=r} f(z)$, and let $x = H(Y)$ denote the inverse function of $Y = \log M(x)$, then $|a_n|^{1/n} H(n) \leq e$ $(n = 1, 2, 3, \cdots)$.*

Lemma 3 [4, p. 13]. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be any entire function of lower order λ $(0 \leq \lambda \leq \infty)$ and let $\nu(r)$ denote the central index, then*

$$\liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda.$$

Lemma 4 [14, p. 24]. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be any entire function of order ρ $(0 < \rho < \infty)$, lower order λ $(0 \leq \lambda \leq \rho < \infty)$, lower type t and type T $(0 \leq t \leq T \leq \infty)$, then*

$$\liminf_{r\to\infty} \frac{\nu(r)}{r^\rho} \begin{matrix} \leq \lambda T, \\ \\ \leq \rho t. \end{matrix}$$

Lemma 5. *Let $f(z)$ be any entire function. Then $D^{k-1} f(z)$ is univalent in a disk of radius R if*

$$\sum_{j=1}^{\infty} (j+1) \frac{|a_{k+j}|}{|a_k|} \frac{e_{j+1} e_k R^j}{e_{j+k} e_1} < 1.$$

Proof. It is clear from the definition that for $Z_1 \neq Z_2$ and $|Z_1| < R, |Z_2| < R$,

$$\left| \frac{D^{k-1} f(z_1) - D^{k-1} f(z_2)}{z_1 - z_2} \right| = \left| \sum_{j=1}^{\infty} \frac{e_j}{e_{k+j-1}} a_{k+j-1} \frac{(z_1^j - z_2^j)}{(z_1 - z_2)} \right|$$

$$\geq \frac{|a_k| e_1}{e_k} - \sum_{j=2}^{\infty} j R^{j-1} \frac{|a_{k+j-1}| e_j}{e_{j+k-1}}.$$

If $|a_k| e_1/e_k > \sum_{j=2}^{\infty} jR^{j-1} |a_{j+k-1}| e_j/e_{j+k-1}$, then clearly $D^{k-1}f(z)$ is univalent for $|Z| < R$, hence the lemma.

3. Zero-free disks for $D^k f(z)$.

Theorem 1. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function, and let ψ_k denote the absolute value of the zero Z_k of $D^k f(z)$ which is nearest to the origin, then denoting by $x = H(Y)$ the inverse function of $Y = \log M(x)$, and with the assumption* (1), *we have*

$$\limsup_{k\to\infty} \frac{d_{k+1}\psi_k}{H(K)} \geq \frac{1}{2e}. \tag{2}$$

Proof. From the definition of $D^k f(z)$ along with Lemma 1, we have, for all those values of k for which Lemma 1 is valid,

$$\begin{aligned} \left| \frac{D^k f(z)}{D^k f(0)} - 1 \right| &\leq \sum_{j=1}^{\infty} \frac{e_k e_j}{e_{k+j}} \frac{|a_{k+j}|}{|a_k|} |z|^j \\ &\leq \sum_{j=1}^{\infty} \frac{d_{k+1} d_{k+2} \cdots d_{k+j}}{d_1 d_2 \cdots d_j} |a_k|^{j/k} |z|^j \end{aligned} \tag{3}$$

$$\leq \sum_{j=1}^{\infty} d_{k+1}^j |a_k|^{j/k} R^j = \frac{1}{1 - |a_k|^{1/k} d_{k+1} R} - 1 \tag{4}$$

by choosing $|a_k|^{1/k} d_{k+1} R < 1$.

Now it is clear from (4) that if $1/(1 - |a_k|^{1/k} d_{k+1} R) - 1 < 1$ then $D^k f(z) \neq 0$ for $|z| \leq R$.

In other words if $R < 1/2 |a_k|^{1/k} d_{k+1}$, then $D^k f(z) \neq 0$ for $|z| \leq R$. That is

$$\psi_k \geq \frac{1}{2 |a_k|^{1/k} d_{k+1}}. \tag{5}$$

Now the required result (1) follows from (5) and Lemma 2.

Remarks. We have from (2) for any entire function $\limsup_{k\to\infty} d_{k+1}\psi_k = \infty$. For $d_k = k$, we have the result of Erdős and Rényi [9, (47)] as a particular case.

Corollary 1. *Let $C = C_k(f)$ denote the absolute value of the zero $|z_k|$ of $\sigma^k f(z)$ (kth normalized remainder), which is closest to the origin, then for any entire function $\limsup_{k\to\infty} C_k = \infty$.*

Proof. This follows easily from (2) by choosing $d_k = 1$.

Theorem 2. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of lower order*

λ $(0 \leq \lambda \leq \infty)$. *If* ψ_k *is defined as in Theorem 1, then*

$$\limsup_{k\to\infty} \frac{\log \psi_k d_{k+1}}{\log_k} \geq \frac{1}{\lambda}.$$

Proof. As usual

$$\left|\frac{D^k f(z)}{D^k f(0)} - 1\right| \leq \sum_{j=1}^{\infty} \frac{e_k e_j}{e_{k+j}} \frac{|a_{k+j}|}{|a_k|} |z|^j.$$

If $f(z)$ is entire, then the radius of convergence $C = \infty$, that is, $|a_n|^{1/n} \to 0$, and thus one can find for *any* $B > 0$ $(B < C)$ an infinity of values of k for which $|a_{k+j}| < |a_k| B^{-j}$ $(j = 1, 2, 3, \cdots)$. Now by substituting this in the above inequality, we have

$$\left|\frac{D^k f(z)}{D^k f(0)} - 1\right| \leq \sum_{j=1}^{\infty} \frac{e_k e_j}{e_{k+j}} \frac{|z|^j}{B^j} \leq \sum_{j=1}^{\infty} d_{k+1}^j |z|^j B^{-j}.$$

We choose here $d_{k+1}|z| < B$, so that we have from (6)

$$\left|\frac{D^k f(z)}{D^k f(0)} - 1\right| \leq \sum_{j=1}^{\infty} d_{k+1}^j |z|^j B^{-j} = \frac{1}{1 - d_{k+1}R/B} - 1. \tag{7}$$

As earlier it is clear, from (7), $D^k f(z) \neq 0$, for $|z| \leq R$ if $R < B/2d_{k+1}$, that is, $\psi_k \geq B/2d_{k+1}$.

Now we choose here $B = r$, $k = \nu(r)$ the central index of f, and clearly $C > r$. Then

$$\frac{\log r}{\log \nu(r)} \leq \frac{\log d_{k+1}\psi_k}{\log \nu(r)} + \frac{\log 2}{\log \nu(r)}.$$

This inequality along with Lemma 3 gives us the required result.

Remarks. If $\lambda < \alpha$ and $d_k = k^{1/\alpha}$ $(\alpha > 0)$, then from Theorem 2 we have $\limsup_{k\to\infty} \psi_k = \infty$. From this it is clear that, if α is very large, λ can be very large, thus we get even for large λ, very large zero free disks tending to infinity. In fact $\limsup_{k\to\infty} \psi_k = \infty$ implies that anywhere in the plane for every disk of arbitrarily large radius, there exist infinitely many k where $D^k f(z) \neq 0$. On the other hand, in the work of Ålander [2], $r_k \to 0$ as $\rho > 1$, and in Boas and Reddy [5], the radius of zero free disks tends to zero when $\rho > 2$. For $d_k = k$, Theorem 2 improves the result of Ålander [2, Theorem 2] and Pólya [13, p. 18]. This also improves another result of Pólya [12,Theorem II] replacing the restriction $\rho < 2/3$ by $\lambda < 2/3$. This suggests that Pólya's hypothetical Theorem A [13, p. 182] may be true for any order ρ as long as the lower order λ is less than 2/3.

Corollary 2. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an entire function of lower order* λ $(0 \le \lambda \le \infty)$. C_k *is defined as in Corollary* 1; *then*

$$\limsup_{k\to\infty} \frac{\log C_k}{\log_k} \ge \frac{1}{\lambda}.$$

Proof. This follows from Theorem 2 by taking $d_k = 1$.

Theorem 3. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an entire function of order* ρ $(0 < \rho < \infty)$, *lower order* $\lambda \ge 0$, *lower type* $t \ge 0$, *type* $T \le \infty$. *Let* ψ_k *denote the absolute value of the zero* Z_k *of* $D^k f(z)$ *which is closest to the origin; then under hypothesis* (1), *we have*

$$\limsup_{k\to\infty} \frac{d_{k+1}\psi_k}{k^{1/\rho}} \begin{cases} \ge \dfrac{1}{2(\lambda T)^{1/\rho}}, \\ \ge \dfrac{1}{2(\rho t)^{1/\rho}}. \end{cases}$$

Proof. The proof of this theorem follows exactly on the same lines as that of Theorem 2, with one difference. We use Lemma 4 instead of Lemma 3; hence we omit the proof.

Corollary 3. *Let* λ, ρ, t *and* T *have the same meaning as in Theorem* 3. *Then,*

$$\limsup_{k\to\infty} \frac{C_k}{k^{1/\rho}} \begin{cases} \ge [2(\lambda T)^{1/\rho}]^{-1}, \\ \ge [2(\rho t)^{1/\rho}]^{-1}. \end{cases}$$

This is a special case of Theorem 3 for $d_k = 1$.

Theorem 4. *Let* $f(z)$ *be an analytic function for* $|z| < r$, *not a polynomial.* ψ_k *denotes the absolute value of the zero of* $D^k f(z)$ *which is nearest to the origin. Then under the assumption* (1), *we have*

$$\limsup_{k\to\infty} d_{k+1}\psi_k \ge \frac{r}{2}.$$

Proof. If $f(z)$ is analytic for $|z| \le r$, then $\limsup_{j\to\infty} |a_j|^{1/j} \le 1/r$; from this we have for any $\epsilon > 0$ that there exist infinitely many k for which $|a_{k+j}/a_k| \le (1+\epsilon)^j r^{-j}$, $j = 0, 1, 2, \cdots$. This along with (3) gives us

$$\left|\frac{D^k f(z)}{D^k f(0)} - 1\right| \le \sum_{j=1}^{\infty} \left\{\frac{d_{k+1}(1+\epsilon)R}{r}\right\}^j = \frac{1}{1 - d_{k+1}(1+\epsilon)R/r} - 1,$$

because of the restriction that $d_{k+1}(1+\epsilon) R < r$. As earlier it is clear from this that

$$D^k f(z) \ne 0 \quad \text{for } |z| \le R, \text{ if } R < r/2d_{k+1}(1+\epsilon), \tag{8}$$

hence $\limsup_{k\to\infty} d_{k+1}\psi_k \geq r/2$, ϵ being arbitrary.

Remarks. For $d_k = 1$, there exist functions $f(z)$ with radius of convergence 1, for which $\liminf_{k\to\infty}\psi_k$ can be determined precisely. It is known [7, (1.2) and Lemma 3] that $\limsup_{k\to\infty}\psi_k = \limsup_{k\to\infty} C_k(f) = 1/P$, where $1.7818 < P < 1.82$. Given any positive small number β, it is possible to construct functions $f(z)$, which are analytic in the unit circle for which $\liminf \psi_k \leq \beta$.

For example, let $f(z) = 1 - \delta z + z^2 - \delta z^3 + z^4 - \delta z^5 + \cdots$, $\delta > 0$; i.e.

$$f(z) = (1 - \delta z)(1 + z^2 + z^4 + \cdots).$$

For this function, it is easy to verify that

$$f(z) = \sigma^2 f(z) = \sigma^4 f(z) = \cdots = \sigma^{2k} f(z), \quad \text{for } k = 0, 1, 2, \cdots.$$

Hence

$$\sigma^{2k} f(z) = (1 - \delta z)(1 + z^2 + z^4 + \cdots), \qquad \delta \geq 1/\beta.$$

From this it follows easily that $\liminf_{k\to\infty}\psi_k = 1/\delta \leq \beta$.

We would like to point out here that there exist entire functions $f(z)$ (cf. [10]) for which zeros of $D^k f(z)$ can be determined with precision.

4. Disks of univalence.

Theorem 5. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be any entire function. U_k denotes the largest disc about the origin where $D^k f(z)$ is univalent ($k = 1, 2, \cdots$). Then denoting by $x = H(Y)$ the inverse function of $Y = \log M(x)$, we have under hypothesis* (1)

$$\limsup_{k\to\infty} \frac{d_{k+1} U_{k-1}}{H(k)} \geq \frac{1}{4e}.$$

Proof. It is known from Lemma 5 that $D^{k-1} f(z)$ is univalent for $|z| < R$, if

$$\sum_{j=1}^{\infty} 2^j \frac{e_j e_k}{e_{j+k}} \left| \frac{a_{j+k}}{a_k} \right| R^j < 1. \tag{9}$$

It is easy to see that the right-hand side of (9) follows from the proof of Theorem 1, by replacing R in (3) by $2R$. Therefore all the calculations based on Theorem 1 are valid for (9) with R replaced by $2R$ in (4). Hence $D^{k-1} f(z)$ is univalent for $|z| < R$ if

$$2R < \frac{1}{2|a_k|^{1/k} d_{k+1}}. \tag{10}$$

Inequality (10) along with Lemma 2 gives us the required result. Similarly we can prove

Theorem 6. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of lower order $\lambda \geq 0$. U_k denotes the radius of the largest disk about the origin where*

$D^k f(z)$ *is univalent. Then under assumption* (1), *we have*

$$\limsup_{k\to\infty} \frac{\log U_{k-1} d_{k+1}}{\log k} \geq \frac{1}{\lambda}.$$

Theorem 7. *Let* $f(z)$ *be an entire function of order* ρ $(0 < \rho < \infty)$, *lower order* λ, *lower type* t *and type* T $(0 \leq t \leq T \leq \infty)$. U_k *denotes the radius of the largest disk about the origin where* $D^k f(z)$ *is univalent. Then under assumption* (1), *we have*

$$\limsup_{k\to\infty} \frac{U_{k-1} d_{k+1}}{k^{1/\rho}} \begin{array}{l} \geq [4(\lambda T)]^{-1}, \\ \\ \geq [4(\rho t)]^{-1}. \end{array}$$

Theorem 8. *Let* $f(z)$ *be regular for* $|z| < r$ *and not a polynomial.* U_k *denotes the radius of the largest disk about the origin where* $D^k f(z)$ *is univalent. Then under assumption* (1),

$$\limsup_{k\to\infty} d_{k+1} U_{k-1} \geq \frac{r}{4}.$$

Remarks on Theorems 5 and 6. For any entire function we have from Theorem 5, $\limsup_{k\to\infty} d_{k+1} U_{k-1} = \infty$. For $d_k = k$, this includes the result of Shah and Trimble as a special case. For the case $d_k = k$, Theorem 6 improves the result of Pólya [13, p. 181]; this includes also the result of Shah and Trimble [15, (3.3)]. For $d_k = k^{1/\alpha}$ $(\alpha > 0)$, one has from Theorem 6 for a sequence of values of k, $U_{k-1} \geq k^{(\lambda+\epsilon)^{-1} - \alpha^{-1}}$, that is $U_{k-1} \uparrow \infty$ if $\lambda < \alpha$. In fact $\limsup_{k\to\infty} U_k = \infty$ implies that for each large disk anywhere in the plane, there exist infinitely many k, for which $D^k f(z)$ is univalent. This result is stronger than the known results in this direction.

Let V_k denote the largest disk about the origin, where $\sigma^k f(z)$ is univalent; then with the help of Theorems 5, 6 and 7, we can replace C_k by $2V_{k-1}$ in Corollaries 1, 2 and 3.

Theorems 5, 6, 7 and 8 can be extended easily to the U_q-radius studied in [8]; the details are left to the reader.

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