

FORMAL TAYLOR SERIES AND COMPLEMENTARY INVARIANT SUBSPACES

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ABSTRACT. A class of operators (which includes the unilateral weighted shifts and the noninvertible bilateral weighted shifts on Hilbert spaces), with the property that every element of the commutant of the operator is canonically associated to a formal Taylor series, is characterized. Let T be one of such operators, then the following result is true: T has no nontrivial complementary invariant subspaces, no roots and no logarithm.

This result can be partially extended to the case when "commutant" is replaced by "double commutant", then: T has no nontrivial complementary hyperinvariant subspaces.

1. Throughout this paper *operator* means *bounded linear operator* (from a complex Banach space into itself) and *subspace* means *closed linear manifold*. Let T be given; \mathfrak{U}_T , \mathfrak{U}_T^a , \mathfrak{U}'_T , \mathfrak{U}''_T will denote the four (weakly closed, containing the identity operator I) algebras canonically associated to T [5]: \mathfrak{U}_T (\mathfrak{U}_T^a , resp.) is the algebra generated by the polynomials (the analytic functions, resp.) in T , \mathfrak{U}'_T (\mathfrak{U}''_T , resp.) is the commutant (the double commutant, resp.) of T . Recall [2, Corollary VI.1.5, p. 477] that \mathfrak{U}_T (\mathfrak{U}_T^a , resp.) is equal to the *strong closure* of the polynomials (the rational functions with poles outside the spectrum $\sigma(T)$ of T , resp.) in T , and $\mathfrak{U}_T \subset \mathfrak{U}_T^a \subset \mathfrak{U}''_T \subset \mathfrak{U}'_T$.

Let T be an operator satisfying the following two properties:

(A) For every $B \in \mathfrak{U}_T$ there exists a net $\{p_\nu(z) = \sum_{k=0}^{N_\nu} c_{\nu k} z^k\}$ ($\nu \in \Lambda$, a directed set) of polynomials such that $\|p_\nu(T)\|$ is uniformly bounded and $p_\nu(T) \rightarrow B$ (strongly), and

(B) If $\{p_\nu(z)\}$ is as above (with $\|p_\nu(T)\|$ uniformly bounded), then $p_\nu(T) \rightarrow 0$ (strongly) if and only if $\Lambda\text{-lim } c_{\nu k} = 0$, $k = 0, 1, 2, \dots$.

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Then to every $B \in \mathfrak{U}_T$ we can associate a *unique formal Taylor series* in T as follows: Let $\{p_\nu\}$ be an arbitrary net satisfying the conditions of (A) and define

$$(1) \quad c_k = \Lambda - \lim c_{\nu k}, \quad k = 0, 1, 2, \dots;$$

it is clear (from (A) and (B)) that the limits actually exist and are independent of the particular net. Then, we can formally write

$$(2) \quad B = \sum_{k=0}^{\infty} c_k T^k.$$

Condition (B) also implies the following property: if $B_1, B_2 \in \mathfrak{U}_T$ and they have the same formal Taylor series, then $B_1 = B_2$.

Lemma 1. *Let T be an operator enjoying the properties (A) and (B), and let $B_1, B_2 \in \mathfrak{U}_T$. Then the formal Taylor series of $B_1 B_2$ ($= B_2 B_1$) is equal to the (formal) product of the formal Taylor series of B_1 and B_2 .*

Proof. Let $\{p_\nu(z)\}$ ($\nu \in \Lambda$), $\{q_\mu(z)\}$ ($\mu \in \Gamma$) (Λ, Γ , two directed sets) be two nets of polynomials such that $p_\nu(T) \rightarrow B_1$, $q_\mu(T) \rightarrow B_2$ (strongly), and $\|p_\nu(T)\| \leq K$, $\|q_\mu(T)\| \leq K$, for all ν and μ . Then, if $x_1, x_2, \dots, x_n \in X$, we have

$$(3) \quad \begin{aligned} \|(p_\nu(T)q_\mu(T) - B_1 B_2)x_j\| &\leq \|p_\nu(T) - B_1\| B_2 x_j\| \\ &\quad + \|(q_\mu(T) - B_2)B_1 x_j\| + 2K\|(p_\nu(T) - B_1)x_j\|. \end{aligned}$$

It is not hard to see that the right side of (3) (and, hence, the left side too) can be made arbitrarily small by taking ν in a cofinal subset of Λ and μ in a cofinal subset of Γ .

Since $\|p_\nu(T)q_\mu(T)\| \leq K^2$, we conclude that $\{p_\nu(T)q_\mu(T)\}$ ($(\nu, \mu) \in \Lambda \times \Gamma$, product order) is a net of polynomials in T converging strongly to $B_1 B_2$ and satisfying the conditions of (A).

Using this particular net and (A), (B), (1) and (2), the result follows. Q.E.D.

A subspace is called *hyperinvariant* (*bi-invariant*, resp.) for T if it is invariant under every operator in \mathfrak{U}'_T (\mathfrak{U}''_T , resp.) (see [1], [5]). It follows from [2, Exercise 3, p. 70] that T has nontrivial complementary invariant subspaces (or "reducing subspaces") if and only if there exists an operator $P \in \mathfrak{U}'_T$, $0 \neq P \neq I$, such that $P^2 = P$, in [5, § 5] it is shown that complementary invariant subspaces are actually bi-invariant and, on the other hand, the existence of nontrivial complementary hyperinvariant subspaces is equivalent to the existence of an operator $P \in \mathfrak{U}''_T$ satisfying the above

conditions. As an application of these results, we have the following:

Theorem 2. *Let T be an operator (on a complex Banach space of dimension larger than zero) enjoying the properties (A) and (B), then:*

(i) *If $\mathfrak{U}_T^a = \mathfrak{U}_T$, then $\sigma(T)$ is connected and contains the origin, and T is logarithmless.*

(ii) *If $\mathfrak{U}_T'' = \mathfrak{U}_T$, then T has no nontrivial complementary hyperinvariant subspaces.*

(iii) *If, moreover, $\mathfrak{U}_T' = \mathfrak{U}_T$, then T is rootless and has no nontrivial complementary invariant subspaces.*

Proof. It is immediate, from Lemma 1 and previous observations, that no $P \in \mathfrak{U}_T$, $0 \neq P \neq I$, can satisfy the equation $P^2 = P$, and no $B \in \mathfrak{U}_T$ can satisfy the equation $B^k = T$, for any integer $k > 1$.

(i) Let $\mathfrak{U}_T^a = \mathfrak{U}_T$. If $\sigma(T)$ is disconnected, then it follows from [2, Chapter VII. 3] that \mathfrak{U}_T^a contains a nontrivial idempotent operator P , a contradiction. Therefore, $\sigma(T)$ is connected.

If $0 \notin \sigma(T)$, then $T^{-1} \in \mathfrak{U}_T^a = \mathfrak{U}_T$, and therefore T^{-1} has a formal Taylor series, but this fact and Lemma 1 imply that the Taylor series of $I = TT^{-1}$ has constant term equal to zero, which is clearly impossible. Hence, $0 \in \sigma(T)$ and therefore T has no logarithm. In fact, for an arbitrary operator L , $0 \notin \sigma(e^L)$ (where $e^L = \lim \sum_{n=0}^N (1/n!)L^n$ ($N \rightarrow \infty$)); therefore, T cannot be expressed as $T = e^L$.

(iii) Let $\mathfrak{U}_T' = \mathfrak{U}_T$ and assume that $B^k = T$, for some integer $k > 1$; clearly, $B \in \mathfrak{U}_T' = \mathfrak{U}_T$, but this is impossible as was remarked above.

(ii) and the second part of (iii) can be proved by using the same argument as in the proof of (i) (first part). Q.E.D.

Remark. The condition "if" of property (B) is essential to prove those results concerning the nonexistence of complementary invariant subspaces; Lemma 1 and the remaining statements of Theorem 2, as well as the next lemma, only depend on the *existence* of the formal Taylor series. This will be made clear in the next section; now, we shall establish without proof an elementary complement to Theorem 2.

Let $B = \sum_{k=0}^{\infty} c_k T^k \in \mathfrak{U}_T$, and denote by $r(B)$ the radius of convergence of the series $\sum_{k=0}^{\infty} c_k z^k$; then

Lemma 3. *If the operator T satisfies (A), (B) and $\mathfrak{U}_T^a = \mathfrak{U}_T$, then $\sigma(T)$ contains the closed disc of radius $r = \inf \{r(B) : B \in \mathfrak{U}_T\}$ about the origin. Furthermore, the result is false if r is replaced by $r + \epsilon$, for any $\epsilon > 0$.*

2. **Examples.** An operator T on a separable Hilbert space is said to be a *unilateral weighted shift operator* (U.W.S.) if there is an orthonormal basis $\{e_n\}_{n=0}^\infty$ and a bounded sequence of *positive* numbers $\{w_n\}_{n=0}^\infty$ such that $Te_n = w_n e_{n+1}$, $n = 0, 1, 2, \dots$.

Similarly, B is said to be a *bilateral weighted shift operator* (B.W.S.) if there is an orthonormal basis $\{f_n\}_{n=-\infty}^{+\infty}$ and a bounded (two-sided) sequence of positive numbers $\{b_n\}_{n=-\infty}^{+\infty}$ such that $Be_n = b_n e_{n+1}$, $n = 0, \pm 1, \pm 2, \dots$.

For elementary properties of these operators, see [4]. In [6], A. L. Shields and L. J. Wallen proved that $\mathcal{U}'_T = \mathcal{U}_T$ for every U.W.S.; more recently, T. R. Turner used the same technique to prove that $\mathcal{U}'_B = \mathcal{U}_B$, for all *noninvertible* B.W.S. (this corresponds to the case when the sequence $\{b_n\}$ is not bounded below). From the *proofs* of those results, it is not hard to see that a U.W.S. or a noninvertible B.W.S. always satisfies the conditions (A) and (B); therefore, Theorem 2 holds for these operators.

These results have already been proved [3, Corollary 2] by R. Gellar, however, the first result in this direction corresponds to N. Suzuki [7].

Here we present a list of examples showing that the results of Theorem 2 are sharp: let K be a compact subset of the complex plane and let $\mathcal{R}(K)$ be the uniform closure of the rational functions with poles outside K ; $\mathcal{R}(K)$ is a Banach algebra (under the "supremum" norm) and every function $f(z) \in \mathcal{R}(K)$ is continuous on K and analytic in the interior of K . If $T = M_g$ (M_g denotes the operator "multiplication by the function $g(z)$ ") then $\mathcal{U}_T = \{M_f : f \in \mathcal{A}(K)\}$, where $\mathcal{A}(K)$ is the uniform closure of the polynomials, and $\mathcal{U}'_T = \mathcal{U}^*_T = \mathcal{U}'_T = \{M_g : g \in \mathcal{R}(K)\}$, $\sigma(T) = K$.

If K does not disconnect the plane, then it follows from the well-known *theorem of Runge* that $\mathcal{U}_T = \mathcal{U}^*_T$, and conversely.

Example 1. If the origin belongs to the unbounded component of the complement of K , then T does not verify (B). To see this, let D_ϵ be a closed disc of radius $\epsilon > 0$ about 0 such that $D_\epsilon \cap K = \emptyset$; by Runge's theorem there exists a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n(z)$ converges to 1 on K , and to 0 on D_ϵ , the convergence being uniform on $D_\epsilon \cup K$. Hence, $1 = \lim p_n(T)$ ($n \rightarrow \infty$), but (by elementary properties of the analytic functions) the limit of the coefficient c_{nk} of z^k in $p_n(z)$ is equal to zero, for all $k = 0, 1, 2, \dots$.

Similarly, $q_n(T) = 1 - p_n(T) \rightarrow 0$, but the constant term of $q_n(z)$ converges to 1.

Condition (A) is always fulfilled by these operators; in fact, if $M_{f_\nu} \in \mathcal{U}'_T$ and $M_{f_\nu} \rightarrow M_{f_0}$ (strongly), then $\|M_{f_\nu} - M_{f_0}\| \rightarrow 0$ (to see this, apply $M_{f_\nu} - M_{f_0}$ to the constant functions!).

With similar arguments it is possible to prove that if 0 is not in the boundary of the unbounded component of the complement of K , then T satisfies the property "only if" of (B) and it is a rootless and logarithmless operator. T^k ($k > 1$) has the same properties, except that it has a k th root.

Example 2. Let K be the unit circle; then $\mathfrak{U}_T^a \neq \mathfrak{U}_T$ and $0 \notin \sigma(T) = K$, even when the property (B) is completely fulfilled and \mathfrak{U}_T^a does not contain nontrivial idempotents. If K' is the union of K and $\{2\}$, then the condition "if" of (B) is not fulfilled (\mathfrak{U}_T contains two nontrivial idempotents).

Example 3. If K is the union of the circles of radius 1 and 2, then T satisfies (B), $0 \notin \sigma(T)$ and therefore $\mathfrak{U}_T^a \neq \mathfrak{U}_T$; however, $\mathfrak{R}(K)$ contains the characteristic functions of those circles and therefore \mathfrak{U}_T^a contains two nontrivial idempotents.

Example 4. If K is the closure of its interior M , $0 \in M$ and the complement of (K) has exactly one component (the unbounded one!), then T satisfies (A) and (B), and $\mathfrak{U}_T = \mathfrak{U}_T^a = \mathfrak{U}_T'$. Therefore, T verifies Theorem 2.

Example 5. Let K and T be as above and define $X = \mathfrak{R}(K) \oplus \mathfrak{R}(K)$, $B = T \oplus T$ ($B(f, g) = (Tf, Tg)$). Then $\mathfrak{U}_B = \mathfrak{U}_B^a = \mathfrak{U}_B'' \neq \mathfrak{U}_B'$. The subspaces $\mathfrak{R}(K) \oplus \{0\}$ and $\{0\} \oplus \mathfrak{R}(K)$ are invariant and complementary; however, by Theorem 2 (ii), B has no nontrivial complementary hyperinvariant subspaces.

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