QUADRATIC JORDAN ALGEBRAS WHOSE ELEMENTS ARE ALL REGULAR OR NILPOTENT¹

KEVIN McCRIMMON

ABSTRACT. We prove that if J is a quadratic Jordan algebra whose elements are all either regular or nilpotent, and which satisfies a common multiple property (that whenever z is nilpotent and v regular then $\text{Im } U_v \cap \text{Ker } U_z \neq 0$), then modulo the radical R the algebra J/R is either a domain or a form of a Jordan algebra determined by a traceless quadratic form in characteristic 2.

For the background of this problem, see [2] and its bibliography. Earlier results on algebras whose elements were invertible or nilpotent were extended in [3], [4], and [5] to the case where the elements were *regular* or nilpotent. M. S. Montgomery showed that a semisimple ample subalgebra of $H(A, *)$ having Ore common multiple property, all of whose elements were regular or nilpotent, was a domain or contained in Φ_2 with A an order in Φ_2 . (An example due to Martindale [6] shows that some hypothesis like the common multiple property is necessary.)

In the present paper we establish Montgomery's theorem in the setting of quadratic Jordan algebras. We say a Jordan algebra J has *common multiple property* if every element of J is either regular or nilpotent, and $\text{Im } U_v \cap \text{Ker } U_z \neq 0$ whenever v is regular and $z \neq 0$ is nilpotent. A Φ -form of an algebra $\tilde{J}$ over $\tilde{\Phi}$ is a Φ -subalgebra $J \subset \tilde{J}$ such that $\tilde{\Phi}J = \tilde{J}$ (we do not assume $J \otimes_{\Phi} \tilde{\Phi} \rightarrow \tilde{\Phi}J$ is an isomorphism).

Theorem A. *A quadratic Jordan algebra J without properly nilpotent elements has the common multiple property if and only if either*

- (i) J is a domain, or
- (ii) J is a Φ -form of a Jordan algebra $J(Q, 1)$ determined by a traceless nondegenerate quadratic form over a field Ω of characteristic 2.

1. Preliminaries. Throughout this paper we will be concerned with a quadratic Jordan algebra J (with product $U_x y$ quadratic in x and linear in

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y) over a commutative associative ring of scalars Φ . We refer to [1] for all results concerning (quadratic) Jordan algebras. We list here the identities we need to recall involving the U -operators,

- (1) $U_{U(x)y} = U_x U_y U_x$
- (2) $U_{x \circ y} + U_{x^2, y^2} = U_x U_y + U_y U_x + U_{x, y}^2$
- (3) $(U_x y)^2 = U_x U_y x^2$
- (4) $(x \circ y)^2 = U_x y^2 + U_y x^2 + x \circ U_y x$
- (5) $U_x U_{x, y} = V_{x^2} V_{x, y} - V_{x^3, y}$
- (6) $U_{x^2, y} = U_{x, y} V_x - V_y U_x = V_x U_{x, y} - U_x V_y$
- (7) $V_y U_x + U_x V_y = U_{x \circ y, x}$
- (8) $U_{x, y} x = x^2 \circ y$

and involving powers

$$(9) \quad U_{x^n} x^m = x^{2n+m}, \quad U_{x^n} = U_x^n.$$

Here $U_{x, y}$ is the linearization of U_x , $x \circ y$ the linearization of x^2 , and $V_{x, y} z = \{x \ y \ z\} = U_{x, z} y$, $V_x y = x \circ y$.

An element x is a *zero divisor* if $U_x y = 0$ for some $y \neq 0$, is *trivial* (or an *absolute zero divisor*) if $U_x = 0$, and is *regular* if it is not a zero divisor. Thus x is regular iff U_x is injective as an operator. Similarly z is nilpotent iff U_z is nilpotent, v is invertible iff U_v is invertible. An element z is *properly nilpotent* if it is nilpotent in all homotopes $J^{(y)}$, equivalently all $U_z^{(y)} = U_z U_y$ are nilpotent. Note that all trivial elements are properly nilpotent. Although it is not known if nil ideals are properly nilpotent, we do have

Lemma 1. *If a quadratic Jordan algebra J has no properly nilpotent elements, it has no nil ideals.*

Proof. Assume $N \subset J$ is a nil ideal; we first show z^2 is properly nilpotent for any $z \in N$. We know all $(U_z U_x) U_z = U_{U(z)x}$ are nilpotent since $U_z x \in U_N J \subset N$ and all elements of N are nilpotent; but an operator AB is nilpotent iff BA is nilpotent, so $U_z (U_z U_x) = U_z^2 U_x = U_z^{(x)} / z^2$ is nilpotent for all $x \in J$ and z^2 is properly nilpotent.

When N has no properly nilpotent elements this implies $z^2 = 0$ for all

$z \in N$, hence $z \circ w = 0$ for $z, w \in N$ by linearization. Furthermore, $z^2 = 0 \Rightarrow U_z^3 = U_z^2 U_z = 0 \Rightarrow z^3 = 0$ since J has no trivial elements. Now

$$(10) \quad z^2 = z^3 = z \circ w = 0 \text{ implies } U_{U(z)w} = U_z U_w U_z = 0$$

since, by (2), $U_z U_w + U_w U_z + U_{z,w}^2 = U_{z \circ w} + U_{z^2, w}^2 = 0$ and thus

$$\begin{aligned} 0 &= U_z \{U_z U_w + U_w U_z + U_{z,w}^2\} \\ &= U_z^2 U_w + U_z U_w U_z + \{V_{z^2} V_{z,w} - V_{z^3, w}\} U_{z,w} \quad (\text{by (5)}) \\ &= U_z U_w U_z. \end{aligned}$$

Then $U_z U_w U_z U_w = \{U_z U_w\}^2 = 0$ for $z, w \in N$. For arbitrary $x \in J$ we have $\{U_z U_x\}^2 = U_z \{U_x U_z U_x\} = U_z U_w$ for $w = U_x z \in U_J N \subset N$ by (1), so $\{U_z U_x\}^4 = \{U_z U_w\}^2 = 0$. This shows all $U_z U_x$ are nilpotent, and z itself is properly nilpotent.

From the absence of properly nilpotent elements we conclude $z = 0$, so $N = 0$ and J has no nil ideals. ■

We denote the sets of regular and nilpotent elements by

$$R = \{v \in J \mid v \text{ is regular}\}, \quad N = \{z \in J \mid z \text{ is nilpotent}\}.$$

Clearly $R \cap N = \emptyset$: no $z \neq 0$ in N is regular, since if $z^n \neq 0$, $z^{n+2} = 0$ then $U_z(z^n) = z^{n+2} = 0$ (by (9)). We are interested in algebras where $R \cup N = J$.

Two important cases of the common multiple property are

Lemma 2. *If every element of J is invertible or nilpotent, then J has common multiple property. If J has the Ore common multiple property that for all nonzero $a \in J$, $v \in R$ there exists $u \in R$, $b \in J$ with $U_a u = U_v b \neq 0$, then J has the common multiple property.*

Proof. When the regular elements are invertible, the common multiple property is trivial: if v is invertible and z nilpotent then $\text{Ker } U_z \neq 0$ and $\text{Im } U_v = J$. If the Ore common multiple property holds and $v \in R$, $z \in N$ with $z^n = a \neq 0$, $z^{n+1} = 0$ then from the existence of u , b as above we have $U_v b = U_a u \neq 0$ in $\text{Im } U_v$ and also in $\text{Ker } U_z$ (since $U_z U_{z^n} u = U_{z^{n+1}} u = 0$) by (9). ■

2. Passage to algebras over a field. Recall the *centroid* $\Gamma(J)$ of a Jordan algebra J is the set of operators T in $\text{End}(J)$ satisfying

$$(11) \quad TU_x = U_x T, \quad TV_x = V_x T, \quad U_{Tx} = T^2 U_x, \quad V_{Tx} = TV_x$$

for all elements x . An easy argument establishes

Proposition 3. *If J is an algebra over its centroid Φ with common multiple property without properly nilpotent elements, and all its elements are regular or nilpotent, then Φ is an integral domain and J is imbedded as a Φ -form of its central closure $\tilde{J} = J_{\tilde{\Phi}}$ which is of the same type over the quotient field $\tilde{\Phi}$ of Φ . ■*

By passing to $\tilde{J}$, we may always assume we are dealing with an algebra over a field. (We do not claim $\tilde{\Phi}$ is the centroid of $\tilde{J}$, though this is true if J is finitely generated as an algebra.) In particular, J has a well-defined characteristic.

3. Proof that x^2 lies in the center. First we investigate how close N is to being an ideal.

Lemma 4. *If $J = N \cup R$ then always $U_N \hat{J} \subset N$ ($\hat{J}$ the unital hull of J), but $U_J N \subset N$ iff J has the common multiple property.*

Proof. To see $U_z a \in J$ belongs to N for any $a \in \hat{J}$, we need only show it is not regular; but if U_z kills $b \neq 0$ then so does $U_{U(z)a} = U_z U_a U_z$ by (1).

Consider $U_a z$ for $a \in J, z \in N$; by the above this automatically belongs to N when $a \in N$, so we assume $a \in R$ is regular. If the common multiple property holds there is some $U_a b \neq 0$ in $\text{Im } U_a \cap \text{Ker } U_z$, so $U_{U(a)z}$ kills $b \neq 0$: $U_{U(a)z} b = U_a U_z (U_a b) \in U_a U_z (\text{Ker } U_z) = 0$. Thus $U_a z$ is not regular and so belongs to N . Conversely, if $U_a z \in N$ is not regular it must kill some $b \neq 0$, $0 = U_{U(a)z} b = U_a U_z (U_a b)$; since a is regular we can cancel U_a to obtain $U_z (U_a b) = 0$, and $U_a b$ lies in $\text{Im } U_a \cap \text{Ker } U_z$ (it is nonzero since $b \neq 0$ and a is regular). ■

This provides further justification for our definition of common multiple property. For convenience of notation we say J is a *CMP algebra* if $J = N \cup R$ has common multiple property but no properly nilpotent elements.

Lemma 5. *If J is a CMP algebra then $z^2 = U_z x^2 = U_z (x \circ y) = 2U_z w = 0$ for all $z, w \in N$ and $x, y \in J$.*

Proof. By Lemma 4 we have $U_z y \in N$ for all $z \in N$ and $y \in J$, so by

(1) $U_{U(z)y} = U_z U_y U_z$ is nilpotent, so (as in Lemma 1) also $U_z(U_z U_y) = U_z^2 U_y = U_z^{(y)}$ is nilpotent, and z^2 is properly nilpotent. By hypothesis $z^2 = 0$.

From $z^2 = 0$ for all $z \in N$ and the fact that $U_x z \in N$ for $x \in J$, $z \in N$ by Lemma 4 (using the common multiple property), we get $0 = (U_x z)^2 = U_x U_z x^2$ by (3). If $x \in N$ trivially $x^2 = 0$ and $U_z x^2 = 0$; if $x \in R$ then $U_x U_z x^2 = 0$ implies $U_z x^2 = 0$. In either case $U_z x^2 = 0$ follows from $N^2 = 0$. Linearization gives $U_z(x \circ y) = 0$.

Since $w \in N$ is nilpotent, $1 - w \in \hat{J}$ is invertible. If $z \in N$ then $U_{1-w}z$ must be in N , too, because it is not regular: if $U_z q = 0$ then $U_{1-w}^{-1} q \in J$ is killed by $U_{1-w}z$,

$$U_{U(1-w)z}(U_{1-w}^{-1}q) = U_{1-w}U_zU_{1-w}(U_{1-w}^{-1}q) = U_{1-w}U_zq = 0.$$

Therefore $(U_{1-w}z)^2 = 0$ and

$$\begin{aligned} 0 &= U_{1-w}^{-1}(U_{1-w}z)^2 = U_{1-w}^{-1}U_{1-w}U_z(1-w)^2 = U_z(1-2w+w^2) \\ &= z^2 - 2U_zw + U_zw^2 = -2U_zw \quad (\text{by (3)}). \quad \blacksquare \end{aligned}$$

Proposition 6. *If J is a CMP algebra of characteristic $\neq 2$ then $N = 0$ and J is a domain, while in characteristic 2 the span Z of all nilpotent elements is an ideal which is either zero (and $J = R$ is a domain) or contains a regular element ($Z \cap R \neq 0$); in the latter case Z has no trivial elements.*

Proof. In characteristic $\neq 2$ we will show N is an ideal; by Lemma 1 an algebra without properly nilpotent elements cannot contain a nil ideal, so $N = 0$ and $J = R$ is a domain. First, N is linear: $N + N \subset N$ since for $z, w \in N$ we have

$$\begin{aligned} (z+w)^3 &= U_{z+w}(z+w) = \{U_z + U_{z,w} + U_w\}(z+w) \\ &= U_{z,w}(z+w) \quad (2U_NN = 0 \text{ in Lemma 6} \Rightarrow U_NN = 0 \text{ in char } \neq 2) \\ &= z^2 \circ w + z \circ w^2 \quad (\text{by (8)}) \\ &= 0 \end{aligned}$$

and $z+w$ is in N . Thus already $(z+w)^2 = z^2 + z \circ w + w^2 = z \circ w$ must vanish, and $N \circ N = 0$. We know N is a strict inner ideal, $U_N \hat{J} \subset N$ by

Lemma 4. It is an outer ideal since $U_J N \subset N$ by Lemma 4 and $V_J N \subset N$ by $(x \circ z)^2 = U_x z^2 + U_z x^2 + z \circ U_x z$ (using (4)) $= z \circ U_x z$ (by Lemma 5) $= z \circ w = 0$ by the above since $w = U_x z \in N$. Thus N is an ideal in characteristic $\neq 2$.

When the characteristic is 2 we have $U_z a^2 = 0$ for all $a = \alpha 1 + x$ in the unital hull $\hat{J}$ ($\alpha \in \Phi, x \in J$) since

$$U_z a^2 = U_z (\alpha^2 1 + 2\alpha x + x^2) = U_z (\alpha^2 1 + x^2) = \alpha^2 z^2 + U_z x^2 = 0$$

from Lemma 5. Then also $(U_a z)^2 = U_a U_z a^2 = 0$ (by (3)) implies $U_a z \in N$ for any $a \in \hat{J}$. This gives us $U_{\hat{N}} \hat{J}^2 = 0, U_{\hat{N}} \hat{J} \subset N, U_{\hat{N}} \hat{J} \subset N$.

It is not quite true that the nilpotent elements form an ideal in characteristic 2, but at least they span an ideal Z . Z consists of all $\sum z_i$ for $z_i \in N$; since $\alpha z \in N$ when $\alpha \in \Phi, z \in N$, Z is clearly a linear subspace. It is an outer ideal since $U_{\hat{J}} N \subset N$ implies $U_{\hat{J}} Z \subset Z$ and $V_{\hat{J}} N \subset Z$ implies $V_{\hat{J}} Z \subset Z$ (to see $V_{\hat{J}} N \subset Z$: $x \circ z = U_{1+x} z - U_x z - z \in N - N - N$ by the previous paragraph). Thus Z will be an inner ideal if $U_z \hat{J} \subset Z$ for the spanning set of elements $z \in N$, which holds since $U_z \hat{J} \subset N \subset Z$ by the above.

If J has no nonzero nil ideals (e.g. if no properly nilpotent elements) then either $Z = 0$ or else $Z \not\subset N$ is not nil and $Z \cap R \neq 0$.

When $v \in Z$ is regular, any element $z \in Z$ which is trivial in Z has $U_v z$ trivial in all of J : $U_z (U_v J) \subset U_z Z = 0$ implies $0 = U_v U_z U_v J = U_{U(v)z} J$ by (1). Since J has no trivial elements, $U_v z = 0$, and since v is regular this forces $z = 0$. Thus Z has no trivial elements either. ■

A proper CMP algebra is one which is not a domain; all such algebras have characteristic 2 with Z containing regular elements but no trivial elements.

We would like to show x^2 is in the "center"; if $x^2 = Q(x)1$ then $x^2 \circ y = Q(x)1 \circ y = 2Q(x)y = 0$ in characteristic 2, so as a preliminary we show $x^2 \circ y = 0$.

Lemma 7. If J is a proper CMP algebra then $V_{x^2} = 0$ for all $x \in J$.

Proof. First we show that if $U_z a = 0$ for $z \in N, a \in J$ then $w = V_z a$ is in N . We have $w^2 = (z \circ a)^2 = U_z a^2 + U_a z^2 + a \circ U_z a = 0$ (using (4)) since $U_z a^2 = z^2 = 0$ by Lemma 5 and $U_z a = 0$ by hypothesis. Thus $w \in N$.

Next we show that if $z, w = z \circ a$ are in N then $w = 0$. Indeed, if $q \in N$ then $U_q w = U_q (z \circ a) = 0$ (Lemma 5) implies $V_q w \in N$ by the previous paragraph, so $(w + q)^2 = w^2 + w \circ q + q^2 = w \circ q \in N$ implies $w + q$ is not regu-

lar, in which case already $w + q \in N$ has $(w + q)^2 = 0$ and $w \circ q = 0$. Since the $q \in N$ span Z we have $w \circ Z = 0$. But once $w^2 = w^3 = w \circ Z = 0$ we saw in (10) $U_{U(w)z} = U_w U_z U_w = 0$ for all $z \in Z$; then all $U_w z$ are trivial in J , so all $U_w z = 0$ and w is trivial in Z , whence $w = 0$ by Proposition 6. Thus $w = z \circ a = 0$, and we have shown

$$U_z a = 0 \Rightarrow V_z a = 0 \quad (z \in N, a \in J).$$

Since $U_z a = 0$ for $a = x^2$ or $x \circ y$ by Lemma 5 we have $z \circ x^2 = z \circ (x \circ y) = 0$ for all $z \in N$, $x, y \in J$. But then $N \circ x^2 = 0$ implies $x^2 \circ Z = 0$, too. To see $x^2 \circ J = 0$, choose a regular $v \in Z$ (Proposition 6) and note

$$\begin{aligned} U_v(V_{x^2}J) &= \{U_{v \circ x^2, v} - V_{x^2}U_v\}J \quad (\text{by (7)}) \\ &= -V_{x^2}(U_v J) \quad (Z \circ x^2 = 0) \\ &\subset V_{x^2}Z \quad (\text{since } v \in Z \text{ and } Z \text{ is an ideal}) \\ &= 0. \end{aligned}$$

Since v is regular we can cancel U_v to obtain $V_{x^2}J = 0$. ■

Once $V_{x^2} = 0$ we can show the U_{x^2} belong to the centroid.

Lemma 8. *If $V_{x^2} = 0$ for all $x \in J$ of characteristic 2 then all U_{x^2} belong to the centroid $\Gamma(J)$.*

Proof. It is enough if $T = U_{x^2}$ commutes with all U_a and V_a , since then

$$U_{Ta} = U_{U(x^2)a} = U_{x^2}U_aU_{x^2} = TU_aT$$

and

$$V_{Ta}(b) = Ta \circ b = V_bTa = TV_ba = TV_a(b)$$

show $U_{Ta} = T^2U_a$ and $V_{Ta} = TV_a$.

It is easy to show T commutes with V_a :

$$\begin{aligned} TV_a - V_aT &= U_{x^2}V_a + V_aU_{x^2} \quad (\text{char } 2) \\ &= U_{x^2 \circ a, x^2} \quad (\text{by (7)}) \\ &= 0 \quad \text{since } x^2 \circ J = 0. \end{aligned}$$

Then

$$\begin{aligned}
TU_a - U_aT &= U_{x^2}U_a + U_aU_{x^2} = U_{x^2 \circ a} + U_{x^4, a^2} - U_{x^2, a}^2 \quad (\text{by (2)}) \\
&= U_{x^2, a}^2 \quad (x^2 \circ J = 0 \text{ and } U_{x^4, a^2} = U_{x^2, a^2}V_{x^2} - V_{a^2}U_{x^2}) \\
&\quad \text{by (6) where } V_{x^2} = V_{a^2} = 0 \text{ by hypothesis)} \\
&= \{U_{x, a}V_x - V_aU_x\}\{V_xU_{x, a} - U_xV_a\} \quad (\text{by (6)}) \\
&= U_{x, a}V_x^2U_{x, a} + V_aU_{x^2}V_a - \{U_{x, a}V_xU_xV_a + V_aU_xV_xU_{x, a}\} \\
&= U_{x, a}V_x^2U_{x, a} + V_a^2U_{x^2} - \{U_{x, a}U_xV_xV_a + V_aV_xU_xU_{x, a}\} \\
&\quad (U_{x^2} \text{ commutes with } V_a \text{ by the above}) \\
&= 0
\end{aligned}$$

because $V_zV_w = 0$ for all z, w in J ($V_zV_wy = z \circ (w \circ y) = V_{y \circ w}z$ and $V_{y^2} = 0$ for all y implies $V_{y \circ w} = 0$ for all y, w by linearization). Thus T also commutes with U_a . ■

4. Proof of the main theorem. It is not hard to check the "if" assertion in Theorem A; the "only if" part asserts that a CMP algebra is either a domain or a form of a degree 2 algebra, equivalently that every proper CMP algebra J is a form of some $J(Q, 1)$.

If J is a proper CMP algebra, so is its central closure $\tilde{J}$ by Proposition 3. Moreover, since $U_{x^2} \in \Gamma(J) = \Phi I$ by Lemmas 7 and 8, the same holds for the central closure: each x in $\tilde{J}$ has the form $x = \gamma^{-1}y$ for $\gamma \in \Phi, y \in J$, so $U_{x^2} = U_{\gamma^{-2}y^2} = \gamma^{-4}U_{y^2} \in \gamma^{-4}\Phi I \subset \tilde{\Phi}I$. But this means each element of J is nilpotent or invertible: the operator U_x is nilpotent or invertible since $U_x^2 = U_{x^2}$ is zero or an invertible element of $\tilde{\Phi}I$.

At this point we could refer to [2] to conclude our $\tilde{J}$ (hence J too) is a form of a $J(Q, 1)$. However, we prefer to indicate a direct proof of the nilpotent-or-invertible case which avoids Osborn's capacity two theorem.

If all elements of $\tilde{J}$ are invertible or nilpotent, and not all are nilpotent, then $\tilde{J}$ must have a unit 1. Then $U_{x^2} \in \tilde{\Phi}I$ implies $x^4 = U_{x^2}1 \in \tilde{\Phi}1$. The relation $V_{x^2} = 0$ also carries over from J to $\tilde{J}$, so it suffices to replace J by $\tilde{J}$ and prove

Theorem B. *If J is a unital quadratic Jordan algebra over a field Φ containing no nil ideals and satisfying $x^2 \circ J = 0, x^4 \in \Phi 1$ for all $x \in J$, then J*

is a form of a Jordan algebra $J(Q, 1)$ determined by a traceless quadratic form.

Proof. Let Ω be the algebraic closure of Φ . The scalar extension J_Ω inherits the properties $x^2 \circ J_\Omega = 0$, $x^4 \in \Omega 1$.

However, J_Ω may have grown a nil ideal, so we divide out by the nil radical N to get $J \subset \bar{J} = J_\Omega/N$. Replacing J by $\bar{J}$, we may assume the base field is algebraically closed.

In the algebraically closed case we claim actually $x^2 \in \Omega 1$. If $x^4 = \gamma 1 \in \Omega 1$ then $z = x^2 + \gamma^{1/2} 1$ satisfies $V_z = z^2 = 0$, so by (10), $z = 0$ and $x^2 = \gamma^{1/2} 1 = Q(x)1$ where Q is a quadratic form on J .

Then $x^3 = Q(x)x$, too, so J has the same square and cube as $J(Q, 1)$, and consequently $J = J(Q, 1)$. ■

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE,
VIRGINIA 22903