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PROOF OF THE GELFAND-KIRILLOV CONJECTURE FOR SOLVABLE LIE ALGEBRAS

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ABSTRACT. Let g be a solvable algebraic Lie algebra over the complex numbers C. It is shown that the quotient field of the enveloping algebra of g is isomorphic to one of the standard fields $D_{n,k}$, being defined as the quotient field of the Weyl algebra of degree n over C extended by k indeterminates. This proves the Gelfand-Kirillov conjecture for g solvable.

1. **Introduction.** Let g be an algebraic Lie algebra over the complex numbers $\mathbf{C}$. Let Ug denote the enveloping algebra of g and Dg the quotient field of Ug. With n, k nonnegative integers, let $A_{n,k}$ denote the Weyl algebra of degree n over $\mathbf{C}$ extended by k indeterminates [1]. Let $D_{n,k}$ denote the quotient field of $A_{n,k}$. It has been conjectured [1]–[3] that Dg is isomorphic to $D_{n,k}$ for suitable n, k. This has been demonstrated for g nilpotent and for g semisimple if either $g = sl(r)$ [2] or Dg is given an extended centre [3]. Here we prove the conjecture for g solvable and algebraic. The analysis is based on the work of Nghiêm [4] and the theorem proved below.

2. **A transcendence theorem.** Let g be a finite dimensional Lie algebra over $\mathbf{C}$ and A an arbitrary commutative subalgebra of Ug. Let $\mathrm{Dim}_{\mathbf{C}}$ denote the dimensionality introduced by Gelfand and Kirillov [1]. We have shown [5, Theorem 1.1] that

$$\mathrm{Dim}_{\mathbf{C}} A \leq \dim g - \tfrac{1}{2} \dim \Omega, \tag{2.1}$$

where Ω is an orbit of maximal dimension in the dual g^* of g [1].

Recall that if g is algebraic, then in particular $g \subset gl(V)$ for some finite dimensional vector space V over $\mathbf{C}$. $g \subset gl(V)$ is said to be almost algebraic [6, p. 98] given that the semisimple and nilpotent parts of every

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$X \in g$ belong to g. g is almost algebraic if it is algebraic [7, Lemma 2, §3]. The proofs of the following two lemmas are straightforward (cf. [8, in particular, Theorem 4]).

Lemma 2.1. *Let $g \subset gl(V)$. Given $X \in g$ semisimple (resp. nilpotent) then $\operatorname{ad}_g X$ is semisimple (resp. nilpotent).*

Lemma 2.2. *Let g be solvable and almost algebraic. Then $g = g_1 \oplus g_2$ where g_1 (resp. g_2) is the commutative subalgebra (resp. nilpotent ideal) of g spanned by the semisimple (resp. nilpotent) elements of g.*

Proof. Let g_1 (resp. g_2) denote a maximal abelian subalgebra of semisimple (resp. the set of all nilpotent) elements of g. Certainly $g_1 \cap g_2 = \{0\}$. By Lie's theorem, it follows that g_2 is a linear space and $g_2 \supset [g, g]$, so g_2 is an ideal. Set $a = g_1 \oplus g_2$. a is an invariant subspace of $\operatorname{ad} g_1$, so by Lemma 2.1 and the choice of g_1, a is complemented in g. That is $g = a \oplus b$, with $[g_1, b] \subset b$. Yet $b \cap g_2 = \{0\}$, so $[g_1, b] = \{0\}$. Since g is almost algebraic, we may write for each $X \in b$: $X = Y + Z$; $Y, Z \in g$, where Y, Z are the semisimple and nilpotent parts of X. We have $[g_1, Y] = \{0\}$, so $Y = 0$, by the maximality of g_1. Then $Z = 0$, since $b \cap g_2 = \{0\}$. Hence $b = \{0\}$, and $a = g$ as required.

Given $g \subset gl(V)$, let $S(V)$ denote the symmetric algebra over V and $K(V)$ its quotient field. Define the action of g on $S(V)$ by derivation and on $K(V)$ by the rule $X\xi = -a^{-2}(Xa)b + a^{-1}(Xb)$ where $\xi = a^{-1}b \in K(V), a, b \in S(V)$. More generally we wish to consider the possibility of there being certain additional algebraic relations in $S(V)$. Let I denote the (two-sided) ideal generated by finitely many g annulled elements of $K(V)$. Assume I prime. Set $S = S(V)/I$ and K its quotient field. Define the subfield K_0 of K annihilated by g through

$$K_0 = \{\xi \in K: X\xi = 0, \text{ for all } X \in g\}.$$

Let $\deg K$ denote the degree of transcendence of K.

Theorem 2.3. *Suppose $g \subset gl(V)$ is solvable and almost algebraic. Define K, K_0 as above. Then K is a pure transcendental extension of K_0. Further, given g nilpotent, $\deg K - \deg K_0 \leq \dim g$.*

Proof. Let $g = g_1 \oplus g_2$ be the decomposition of g defined in the conclusion of Lemma 2.2. Set $T = \{a \in S: g_2 a = 0\}$. Then $gT \subset T$, since g_2 is an ideal. Let L denote the quotient field of T and L_0 the subfield of

L annihilated by g_1. Then $K_0 = L_0$. Indeed $K_0 \supset L_0$ is immediate. On the other hand given $\xi \in K_0$, write $\xi = a^{-1}b$: $a, b \in S$. Suppose that there exists a $Z \in g_2$ such that $Za \neq 0$. Since $Z\xi = 0$, we obtain $\xi = (Za)^{-1}Zb$. Since g_2 is nilpotent it follows that we may write $\xi = c^{-1}d$: $c, d \in T$. Hence $K_0 \subset L_0$ and so $K_0 = L_0$ as required.

Next show that L is a pure transcendental extension of L_0. Let $S^i \subset S$ denote the linear subspace of homogeneous polynomials of degree i. Set $T^i = T \cap S^i$. For each i, S^i and T^i are finite dimensional, g invariant and respectively define a direct sum decomposition of S and T.

Set $\dim g_1 = r$, $\dim g_2 = s$. Let $\{Y_i: i = 1, 2, \cdots, r\}$ be a basis for g_1. Let spec(S) (resp. spec(T)) be the set of all r-tuples $\boldsymbol{\lambda}$ such that

$$Y_i\xi = \boldsymbol{\lambda}_i\xi: \qquad i = 1, 2, \cdots, r,$$

$\xi \in S(\boldsymbol{\lambda})$ (resp. $T(\boldsymbol{\lambda})$) where $S(\boldsymbol{\lambda})$, $T(\boldsymbol{\lambda})$ denote the corresponding eigensubspaces. These subspaces define a direct sum decomposition of S and T respectively. Define $L(\boldsymbol{\lambda})$, spec(L) analogously. Since I is prime a monomial cannot vanish in S. Hence spec(S) and spec(T) are closed under the addition of r-tuples. We remark that spec$(S) \supset$ spec(T) and that this may be strict inclusion. Further spec(S) is generated over the integers by spec(S^1). Let W be the vector space generated by spec(S^1) over the rationals. Set $t = \dim W \leq \dim V$. Let $\{\boldsymbol{\lambda}_j \in \text{spec}(S^1)\}$ be a basis for W. Let u be the common divisor for the rational coefficients of the remaining $\boldsymbol{\mu}_j \in \text{spec}(S^1)$ expressed in this basis. Then for all $\boldsymbol{\lambda} \in \text{spec}(T)$ we have

$$\boldsymbol{\lambda} = \frac{1}{u}\sum_{j=1}^{t} u_j\boldsymbol{\lambda}_j \tag{2.2}$$

for suitable integers u_j. Let M be the module over the integers generated by spec(T). With respect to (2.2) define a map $\pi: M \longrightarrow \mathbf{Z}^t$, through $\pi(\boldsymbol{\lambda}) = \mathbf{u}$. Then M can be considered as a submodule of $\mathbf{Z}^t$. Hence M is a finitely generated free $\mathbf{Z}$ module. Let $\{\boldsymbol{\Lambda}_\alpha\}$ be a basis for M. Taking products of elements of T with integer exponents gives $M = \text{spec}(L)$, so there exist $y_\alpha \in L(\boldsymbol{\lambda})$ such that $Y_i y_\alpha = \Lambda_{i\alpha}y_\alpha$ for all i, α. Further by definition of $\{\boldsymbol{\Lambda}_\alpha\}$ distinct monomials in the y_α belong to distinct eigensubspaces $L(\boldsymbol{\lambda})$. Hence they are algebraically independent over L_0. Now given $u, v \in T(\boldsymbol{\lambda})$, $uv^{-1} \in L_0$, so $u = bv$ for some $b \in L_0$. Recalling the direct sum decomposition of T, it follows that $L + L_0(y_1, y_2, \cdots, y_t)$: $t \leq \dim V$ as required.

Next show that K is a pure transcendental extension of L. In this it is convenient to set $g_1 = h$, $g_2 = g$. Since g is nilpotent, there exists an upper central series $\{0\} = C_1 \subset C_2 \subset C_3 \subset \cdots \subset C_l = g$, for g [6, p.29]. (Recall that C_i is the maximal ideal in g such that $[g, C_i] \subset C_{i-1}$.) That $[h, C_i] \subset C_i$, follows by induction on i and the relation $[g, [h, C_i]] \subset C_{i-1} + [h, C_{i-1}]$. By definition of h and Lemma 2.1, $\mathrm{ad}_g h$ is a commutative Lie algebra of semisimple elements. Hence there exists a central descending series $g = g_1 \supset g_2 \supset \cdots \supset g_{s+1} = \{0\}$: $s = \dim g$, for g with the property that for each integer k there exists $Z_k \in g_k$, $Z_k \notin g_{k+1}$ such that Z_k is an eigenvector for $\mathrm{ad}_g h$. (Recall also that $\dim g_k = \dim g_{k+1} + 1$ and $[g, g_k] \subset g_{k+1}$, for all k.)

Let L_k denote the subfield of K annihilated by g_k. Then $L = L_1 \subset L_2 \subset \cdots \subset L_{s+1} = K$. Let $\tilde{S}$ be the maximal subalgebra of K on which the elements of g are locally nilpotent derivations. Set $\tilde{S}_k = L_k \cap \tilde{S}$ and $S_k = L_k \cap S$. Certainly $\tilde{S}_k$ contains the subalgebra of K generated by L and S_k. In fact we shall see that it coincides with it.

We show that for each k either $L_{k+1} = L_k$ or $L_{k+1} = L_k(a)$ for some $a \in \tilde{S}_{k+1}$. This will prove the theorem with $\deg K - \deg L \leq \dim g$ as required.

Verify that $Z_i S_j \subset S_j$, $Z_i \tilde{S}_j \subset \tilde{S}_j$ and $Z_i L_j \subset L_j$ for all i, j. We show that L_k is the quotient field of S_k by induction on k. By definition $K = L_{s+1}$ is the quotient field of $S = S_{s+1}$. So, given $\xi \in L_k \subset L_{k+1}$, we may write by the induction hypothesis $\xi = a^{-1}b$: $a, b \in S_{k+1}$ Suppose that $Z_k a \neq 0$, then $Z_k \xi = 0$ gives $\xi = (Z_k a)^{-1} Z_k b$. Then by the local nilpotency of Z_k on S_{k+1} we may choose $a, b \in S_k$.

Finally suppose that $L_{k+1} \neq L_k$. Then there exists $a' \in S_{k+1}$ such that $Z_k a' \neq 0$ so by the local nilpotency of Z_k on S_{k+1}, there exists $a'' \in S_{k+1}$ such that $Z_k a'' = b \in S_k$: $b \neq 0$. Now suppose that $Z_i b \neq 0$, for some i. Then $Z_k(Z_i a'') = [Z_k, Z_i]a'' + Z_i(Z_k a'') = Z_i b \neq 0$, since $[Z_k, Z_i] \in g_{k+1}$ and $a'' \in S_{k+1}$. Hence by the local nilpotency of Z_i on S_k we may choose $a''' \in S_{k+1}$ such that $Z_k a''' = b' \in L$: $b' \neq 0$. Then there exists $a \in \tilde{S}_{k+1}$ such that $Z_k a = 1$. Let $\alpha_n a^n + \cdots + \alpha_0$: $\alpha_i \in L_k$, $\alpha_n \neq 0$, be the minimal polynomial of a in L_k. The relation $Z_k a^i = i a^{i-1}$ contradicts its minimality, so a is transcendental over L_k. Further for any $c \in \tilde{S}_{k+1}$, we have $Z_k^n c = d \in \tilde{S}_k$: $d \neq 0$, for some nonnegative integer n. Then $e = c - (n!)^{-1} a^n d$ satisfies $Z_k^{n-1} e = f \in \tilde{S}_k$. Hence by induction $\tilde{S}_{k+1} = \tilde{S}_k[a]$. Then $L_{k+1} = L_k(a)$, which proves the theorem.

Preserve the notation used in the proof of Theorem 2.3. For each $k = 1, 2, \cdots, s+1$; $\boldsymbol{\lambda} \in \operatorname{spec}(S)$ (resp. $\boldsymbol{\lambda} \in \operatorname{spec}(T)$) set $S_k(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}) \cap S_k$ (resp. $\tilde{S}_k(\boldsymbol{\lambda}) = T(\boldsymbol{\lambda}) \cap \tilde{S}_k$). Recall that by construction $[Y_i, Z_j] = \mu_{ij} Z_j$: $\mu_{ij} \in \mathbf{C}$. Let $\boldsymbol{\mu}_j$ denote the r-tuple with entries μ_{ij}.

Lemma 2.4. *For all* $j, k = 1, 2, \cdots, s+1$,

(1) $Z_j S_k(\boldsymbol{\lambda}) \subset S_k(\boldsymbol{\lambda} + \boldsymbol{\mu}_j)$: *for all* $\boldsymbol{\lambda} \in \operatorname{spec}(S)$,

(2) $Z_j \mathfrak{T}_k(\boldsymbol{\lambda}) \subset \mathfrak{T}_k(\boldsymbol{\lambda} + \boldsymbol{\mu}_j)$: *for all* $\boldsymbol{\lambda} \in \operatorname{spec}(T)$,

(3) $S_k = \bigoplus_{\boldsymbol{\lambda} \in \operatorname{spec}(S)} S_k(\boldsymbol{\lambda})$ *(direct sum).*

Proof. (1) and (2) are clear. (3) holds for $k = s+1$ since $S_{s+1} = S$ and S is a direct sum of its eigensubspaces. Assume (3) holds for all $k > j$. To show it holds for $k = j$ recall that $S_j \subset S_{j+1}$ and apply (1).

Corollary 2.5. *Suppose* $S_{k+1} \neq S_k$ *for some* $k \in (1, 2, \cdots, s)$. *Then there exists* $\boldsymbol{\lambda}_k \in \operatorname{spec}(T)$ *such that* $\tilde{\mathfrak{T}}_{k+1} = \tilde{\mathfrak{T}}_k[a_k]$ *with* $a_k \in \tilde{\mathfrak{T}}_{k+1}(\boldsymbol{\lambda}_k)$.

Proof. Apply Lemma 2.4 to the latter half of the proof of Theorem 2.3, noting that we may choose $a' \in S_{k+1}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \operatorname{spec}(S)$.

The conclusion of Theorem 2.3 shows that we may write $K = K_0(y_1, y_2, \cdots, y_t)$. Assume $t \leq \dim g$; let $\{T_i\}$: $i = 1, 2, \cdots, t$, be a subbasis of a basis for g and denote by B the matrix with entries $T_i y_j$. Recall that $g = g_1 \oplus g_2$ and let $\{Y_i\}$, $\{Z_j\}$ be the bases for g defined in the proof of Theorem 2.3. Assume $\{Y_\alpha\}$: $\alpha = 1, 2, \cdots, \gamma$, $\{Z_\mu\}$: $\mu = 1, 2, \cdots, \rho$ to be subbases on which B is nondegenerate. Further assume that

$$Z_k \in \{Z_\mu\} \quad \text{given} \quad L_{k+1} \neq L_k. \tag{2.3}$$

Following Theorem 2.3 introduce subsets $\{y'_\alpha\}, \{y''_\mu\}$ of y_i such that $L = K_0(y'_1, y'_2, \cdots, y'_{\gamma'})$, $K = L(y''_1, y''_2, \cdots, y''_{\rho'})$. Note that $\gamma + \rho = \gamma' + \rho' = t$. By (2.3) $\rho' \leq \rho$. Since $y'_\alpha \in L$, $B_{\mu\alpha} = 0$ for the given ranges of μ, α. Then $\det B \neq 0$, implies $\rho' \geq \rho$. Hence $\rho' = \rho$ and $\gamma' = \gamma$. Then the $B_{\mu\nu}$ are the entries of a square matrix which is upper triangular, has ones on the diagonal, and its remaining entries lie in $\mathfrak{T}$ which was shown to be identical with $L[y''_1, y''_2, \cdots, y''_\rho]$. Since the $y'_\alpha \in L$, it follows that $\det B \in L$. As was shown, the y_i may be chosen as eigenfunctions of the elements of g_1. Then $B_{\alpha i} = \lambda_{\alpha i} y_i : \lambda_{\alpha i} \in \mathbf{C}$. Let B^{-1} denote the inverse of B. Then $B^{-1}_{\alpha\beta} \in L$, for all α, β, and B^{-1} defines a matrix which is upper triangular, has ones on the diagonal, and its remaining entries lie in $\mathfrak{T}$. Finally $B^{-1}_{\alpha\mu}$ are polynomials in the y''_ν over L with no constant terms.

Given $a \in K$, then

$$Za = b \in \tilde{S} \quad \text{for all } Z \in g_2 \text{ implies that } a \in \tilde{S}. \tag{2.4}$$

Indeed by definition of $\mathfrak{T}$, there exists for each $Z \in g_2$ a positive integer m such that $Z^m b = 0$. But then $Z^{m+1} a = 0$ which implies that $a \in \mathfrak{T}$.

We remark that the above conclusions hold should $\mathbf{C}$ be replaced by an arbitrary algebraically closed commutative field of characteristic zero.

3. Proof of conjecture.

Theorem 3.1 [4, Proposition III.8, Theorem III.9]. *Given g solvable, then*

(1) *there exist pairwise commutative elements $A_\alpha \in Ug$: $\alpha = 1, 2, \cdots, s$, algebraically independent over $\mathbf{C}$. Set $A = \mathbf{C}[A_1, A_2, \cdots, A_s]$ and let K denote its quotient field. The $[g, K] \subset K$, and K is a maximal commutative subfield of Dg;*

(2) *there exist $T_i \in g$: $i = 1, 2, \cdots, t$, algebraically independent over K such that $(1, T_1, T_2, \cdots, T_t)$ is a basis for the extension $\tilde{g}$ of g by K. Further $Dg = D\tilde{g}$ and $[T_i, A_\alpha] = A_{i\alpha} \in K$ where the matrix A with entries $A_{i\alpha}$ is of rank t in K.*

(3) $\dim_{\mathbf{C}} C(Dg) \leq s - t$ *and* $C(Dg) \subset K$, *where C denotes centre.*

Set $K_0 = C(Dg)$. Set $m = \dim g$, $n = \frac{1}{2} \dim \Omega$, where Ω is an orbit of maximal dimension in the dual g^* of g [1].

Theorem 3.2. *Suppose g is solvable and algebraic. Define m, n as above. Then Dg is isomorphic to $D_{n,k}$ with $k = m - 2n$.*

Proof. By [1] (cf. [9, Lemma 7]), $\operatorname{Dim}_{\mathbf{C}} C(Dg) = m - 2n = k$. Since g is algebraic, (2.1) applies to the algebra A defined in (1) above. Hence $2s \leq 2m - 2n = m + k$. Yet $m = \dim g = s + t$, where t is defined in (2). Then by (3), $2s \geq m + k$. Hence $k = s - t$ and $n = t$. Further $t = \deg K - \deg K_0$.

Since $A \in Ug$, the space V generated by $\operatorname{ad} g$ on A is finite dimensional. Further the algebra S generated by V, being a subalgebra of Ug, has no zero divisors. By (1) its quotient field is precisely K which is commutative. Since g is solvable algebraic, it follows by Lemma 2.1 that $\operatorname{ad} g$ considered as a subalgebra of $gl(V)$ is solvable and almost algebraic. Recalling (1), (3) above, Theorem 2.3 applies to show that $K_0(y_1, y_2, \cdots, y_t)$. Further by [10], there exist $z_i \in C(Dg)$, such that $K_0 = \mathbf{C}(z_1, z_2, \cdots, z_k)$. Set

$$B_\alpha = \begin{cases} y_\alpha: & \alpha = 1, 2, \cdots, t, \\ z_{\alpha - t}: & \alpha = t + 1, \cdots, s. \end{cases}$$

Define $B'_{i\alpha} = \Sigma_\beta A_{i\beta} (\partial B_\alpha / \partial A_\beta)$. Clearly $\operatorname{rank} B' = \operatorname{rank} A = t$ in K by (2) above. Further define a $t \times t$ matrix B with entries

$$B_{ij} = [T_i, y_j] = B'_{ij}: \quad i, j = 1, 2, \cdots, t.$$

Then $\det B \neq 0$, since $B'_{i\alpha} = 0$: $\alpha > t$.

Set $x_j = \Sigma_i B^{-1}_{ji} T_i$. By construction

$$[x_i, y_j] = \delta_{ij} 1, \tag{3.2}$$

whereas

$$[y_i, y_j] = 0, \tag{3.3}$$

since $y_i \in K$. Note that (3.2) implies $(\mathrm{ad}\, x_i)u = \partial/\partial y_i(u)$ for all $u \in K$. Then by (3.2) and (3.3): $[[x_i, x_j], y_k] = 0$ for all i, j, k. Hence

$$[[x_i, x_j], A_\alpha] = 0 \tag{3.4}$$

for all $i, j = 1, 2, \cdots, t$, $\alpha = 1, 2, \cdots, s$. Yet $x_i \in \tilde{g}$ so by the maximality of K: $[x_i, x_j] \in K$. Set $[x_i, x_j] = f_{ij}$. By antisymmetry, (3.2) and the Jacobi identity applied to $[[x_i, x_j], x_k]$ we obtain

$$\begin{aligned} f_{ij} + f_{ji} &= 0, \\ \partial_k f_{ij} + \partial_i f_{jk} + \partial_j f_{ki} &= 0, \end{aligned} \tag{3.5}$$

where ∂_k denotes differentiation in y_k. The theorem is proved if we can show that there exist $g_i \in K$, such that

$$f_{ij} = \partial_i g_j - \partial_j g_i, \tag{3.6}$$

for all i, j. For then replacing x_i by $x_i - g_i$, we obtain

$$[x_i, x_j] = 0. \tag{3.7}$$

Then by (2) above the x_i, y_i, z_j: $i = 1, 2, \cdots, t$, $j = 1, 2, \cdots, k$, generate Dg. Recalling that $z_j \in K_0$, it follows by (3.2), (3.3) and (3.7) that Dg is isomorphic to $D_{n,k}$ as required.

Let A_r be the rth generator of K constructed in the recurrence procedure of [4, Lemma III.4]. Set $K_1 = \mathbb{C}$, $K_{r+1} = K_r(A_r)$. Let $\{X_i\}$ be some basis for g. The cobase $\{T_i\}$ is obtained as a subbasis of $\{X_i\}$ by eliminating at each step one X_i for which $\alpha_i \neq 0$ in the relation: $A_r = \Sigma_j \alpha_j X_j$: $\alpha_j \in K_r$. With $g = g_1 \oplus g_2$, choose the bases $\{Y_i\}$, $\{Z_j\}$ for g_1, g_2 described in §2. Should $\alpha_i \neq 0$, for some Y_i, eliminate the Y_i. Otherwise eliminate the Z_j belonging to the smallest j for which $\alpha_j \neq 0$. Let $\{Y_\alpha\}$, $\{Z_\mu\}$ denote the resulting cobase. By construction if Z_k is not in the cobase

$$Z_k = \sum_{\mu > k} \alpha_{k\mu} Z_\mu + \beta_k: \quad \alpha_{k\mu}, \beta_k \in K. \tag{3.8}$$

Suppose $L_{k+1} \neq L_k$. By Theorem 2.3, $L_{k+1} = L_k(a_k)$ where $[Z_k, a_k] = 1$ and $a_k \in L_{k+1}$. This contradicts (3.8). Hence (2.3) holds for $\{Z_\mu\}$. Further the conclusion $\rho' = \rho$ (established in the discussion following (2.3)) implies that

(3.9) $$Z_k \in \{Z_\mu\} \text{ if and only if } L_{k+1} \neq L_k.$$

Recalling the choice of the cobase substitution from (3.8) gives

$$[Y_\alpha, Y_\beta] = 0, \qquad [Y_\alpha, Z_\mu] = C_{\alpha\mu} Z_\mu: \qquad C_{\alpha\mu} \in \mathbf{C},$$

$$[Z_\mu, Z_\nu] = \sum_\lambda \gamma^\lambda_{\mu\nu} Z + \Gamma_{\mu\nu}: \qquad \gamma^\lambda_{\mu\nu}, \Gamma_{\mu\nu} \in K.$$

We show that $\Gamma_{\mu\nu} \in \mathfrak{S}$. By (2.4) and the nilpotency of g_2 it suffices to show that $\gamma^\lambda_{\mu\nu} \in \mathfrak{S}$. Now by (3.9), recalling the definition of the a_k, we have

$$\gamma^\lambda_{\mu\nu} = [[Z_\mu, Z_\nu], a_\lambda] - \sum_{\sigma=1}^{\lambda-1} \gamma^\sigma_{\mu\nu}[Z_\sigma, a_\lambda].$$

Then, using (2.4), induction on λ proves the assertion.

Construct the y_i and the x_i as above. Then the only contribution to $[x_i, x_j]$ comes from $\Gamma_{\mu\nu}$. Recalling (§2) that $B^{-1}_{i\mu} \in \tilde{S}$ we obtain

(3.10) $$[x_i, x_j] = \sum_{\mu\nu} B^{-1}_{i\mu} B^{-1}_{j\nu} \Gamma_{\mu\nu} \in \tilde{S}.$$

Set $x'_\alpha = \Sigma_i B^{-1}_{\alpha i} T_i$, $x''_\mu = \Sigma_i B^{-1}_{\mu i} T_i$. Since by (3.10), the f_{ij} are polynomials in the y''_μ over L, we may integrate (3.5) in the y''_μ to obtain functions $g_i \in S$ such that the $\overline{x}_i = x_i - g_i$, satisfy

(3.11) $$[\overline{x}_i, \overline{x}_\mu] = 0,$$

for all i, μ. This integration is performed as follows. Assume $\rho \geq 1$ and identify y''_ρ with y_t. Set $g_t = 0$ and $g_i = \int_0^{y_t} f_{ti}\, dy_t$, for $1 \leq i < t$. By (3.10) f_{ti} is polynomial in y_t and so $g_i \in \tilde{S}$ for all i. Set $f'_{ij} = f_{ij} - (\partial_i g_j - \partial_j g_i)$. Then $f'_{ij} \in \tilde{S}$ and satisfies (3.5). $f'_{ti} = 0$ by construction, so by (3.5) $f'_{it} = 0$ and $\partial_t f'_{ij} = 0$ for all i, j. Then (3.11) obtains by successive integration in each y''_μ. Note further that g_i as polynomials in the y''_μ over L are chosen to have no constant terms.

We show that the $[\overline{x}_\alpha, \overline{x}_\beta]$, as polynomials in the y''_μ over L, have no constant terms. This property was demonstrated in §2 for the $B^{-1}_{\alpha\mu}$. It also holds for the $[Y_\gamma, g_\alpha]$ by choice of the g_α and because the y''_μ are eigenvectors of $\operatorname{ad} Y_\gamma$ where $(\operatorname{ad} Y_\gamma) L \subset L$ for all γ. It is then sufficient to observe that

$$[\bar{x}_\alpha, \bar{x}_\beta] = \sum_{\mu\nu} B_{\alpha\mu}^{-1} B_{\beta\nu}^{-1} \Gamma_{\mu\nu} + [x_\beta, g_\alpha] - [x_\alpha, g_\beta],$$

where

$$[x_\beta, g_\alpha] = \sum_\gamma B_{\beta\gamma}^{-1} [Y_\gamma, g_\alpha] + \sum_\mu B_{\beta\mu}^{-1} [Z_\mu, g_\alpha].$$

Yet $[\bar{x}_\mu, [\bar{x}_\alpha, \bar{x}_\beta]] = 0$ by (3.11) and the Jacobi identity, so $[\bar{x}_\alpha, \bar{x}_\beta] \in L$ by (3.2). Hence $[\bar{x}_\alpha, \bar{x}_\beta] = 0$. Combined with (3.11) it follows that the required functions g_i exist and the theorem is proved.

Remarks. The theorem evidently fails should $k - \deg K_0$ be an odd integer. Yet given $\deg K_0 = k$, it is sufficient that g be almost algebraic. In this connection see [2, §8] for examples. If g is not almost algebraic the integration of (3.5) may fail in K. For example, let e_{ij} denote the usual canonical basis in $\operatorname{Hom}(\mathbf{C}^5, \mathbf{C}^5)$. Set $a_1 = e_{12} - e_{55}$, $a_2 = e_{23} - e_{44}$, $y_1 = e_{15}$, $y_2 = e_{14}$, $z = e_{13}$. Let $h \subset gl(\mathbf{C}^5)$ be the Lie algebra spanned by these elements. In h we have the bracket relations $[a_1, a_2] = z$, $[a_1, y_1] = y_1$, $[a_2, y_2] = y_2$ and all other brackets vanish. h is evidently solvable, yet not almost algebraic. Set $K = \mathbf{C}(y_1, y_2, z)$. Then $K_0 = \mathbf{C}(z)$, $x_1 = y_1^{-1} a_1$, $x_2 = y_2^{-1} a_2$. Equations (3.1) and (3.2) hold; but $f_{12} = z/y_1 y_2$ and (3.5) does not admit integration in K.

Finally given g nilpotent, it is easy to see that the common divisor of x_i, y_i, z_j lies in $C(Ug)$, so we have incidentally proved [2, Lemma 9].

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