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SEQUENTIAL AND CONDITIONAL COMPACTNESS IN THE DUAL OF A BARRELLED SPACE

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ABSTRACT. Let E be a barrelled locally convex space and suppose $T_{\mathfrak{A}}$ is a topology on the dual E' of E which is admissible for the duality (E, E'). It is shown that each $T_{\mathfrak{A}}$ sequentially compact subset of E' is $T_{\mathfrak{A}}$ conditionally compact.

Let X denote a Banach space and $\tau(X', X)$ the Mackey topology for the dual X' of X. J. Howard [1] used a result of Grothendieck to show that each $\tau(X', X)$ sequentially compact subset of X' is $\tau(X', X)$ conditionally compact. We give a direct proof of a more general result.

Let E denote a locally convex topological vector space. A topology $T_{\mathfrak{A}}$ on the dual E' of E is admissible for the duality (E, E') if it is the topology of uniform convergence on a family $\mathfrak{A}$ of subsets of E satisfying conditions B1–B3 on p. 46 of [4]. Also, $\sigma(E', E)$ denotes the weak star topology on E', so that each admissible topology on E' is finer than $\sigma(E', E)$.

Theorem. *Let E be a barrelled space and suppose $T_{\mathfrak{A}}$ is a topology on E' admissible for (E, E'). Then each $T_{\mathfrak{A}}$ sequentially compact subset of E' is $T_{\mathfrak{A}}$ conditionally compact.*

Proof. By [3, 21.4 (4)], $(E', \sigma(E', E))$ is a boundedly complete space. It follows from [3, 18.4,(4b)] that $(E', T_{\mathfrak{A}})$ is boundedly complete. Let A' be a $T_{\mathfrak{A}}$ sequentially compact subset of E'. Then A' is $T_{\mathfrak{A}}$ countably compact, hence $T_{\mathfrak{A}}$ totally bounded [3, 5.6 (3)]. It follows from [4, Lemma 3, p. 49] that the $T_{\mathfrak{A}}$ closure $\overline{A}'$ of A' is $T_{\mathfrak{A}}$ totally bounded, hence $T_{\mathfrak{A}}$ bounded and closed. Therefore, $\overline{A}'$ is $T_{\mathfrak{A}}$ complete, hence $T_{\mathfrak{A}}$ compact.

Remarks. (1) Since a Banach space X is barrelled and $\tau(X', X)$ is admissible for (X, X'), the result of [1] stated above is a special case of the Theorem.

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(2) As was shown in [1], there are barrelled spaces in which the converse of our result is false.

(3) By [2, Problem 19C], a bornological space E in which the closed convex hull of a compact set is compact has the property that $(E', T_{\mathfrak{a}})$ is complete for certain admissible topologies $T_{\mathfrak{a}}$ (including $\tau(E', E)$). The proof of the Theorem thus holds for such topologies also.

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