σ - BUT NOT O -TAUBERIAN THEOREMS

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ABSTRACT. Let A be a regular matrix summability method. Relations between different Tauberian theorems for A are discussed. We show that most summability methods possess some σ -Tauberian theorem with the property that the corresponding O -Tauberian theorem is not valid.

1. Introduction. We consider conditions of the form

$$(1) \quad u_n = o(d_n)$$

and

$$(2) \quad u_n = O(d_n)$$

where the $d_n \geq 0$ are given constants with $\sum d_n = +\infty$, and the u_n are terms of an arbitrary series with partial sums s_n . (If $d_n = 0$ for some n , both conditions imply $u_n = 0$.) For each regular summability method A there exist d_n for which (1) is a Tauberian condition for A , see [4, p. 82]: A -summability of the sequence s_n and (1) together imply its convergence. Of course, one is interested to have optimally large d_n , so that (1) is a Tauberian condition for A , but $u_n = o(F_n d_n)$ is not, for any choice of $F_n \rightarrow \infty$. In this case the proof of the corresponding Tauberian theorem with (2) (if this theorem holds) may be quite complicated. Hardy, Littlewood, R. Schmidt, Karamata, Beurling, Delange and others have developed sophisticated proof devices, especially for the Abel and Borel methods.

It would be desirable to have a general principle to pass from σ -theorems to O -theorems. Even better would be to describe all d_n for which (1) or (2) is a Tauberian condition for A . This last problem has been solved for the Abel method in [1] and the Borel method in [3]. If $q_0 < q_1 < \dots$ are gap indices for A (for which a gap Tauberian theorem holds), then the necessary and sufficient condition is

$$\sum_{q_j < n < q_{j+1}} d_n = O(1)$$

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for some q_j in case of (1), and

$$\sum_{q_j < n < q_{j+1}} d_n < \epsilon$$

for some properly chosen q_j in case of (2). It would be interesting to have similar results for other methods.

In this paper we show that the principle described above does not exist for arbitrary d_n . For most summability methods there exist constants d_n for which (1) but not (2) is a Tauberian condition. Our considerations are based on ideas developed in [1], connections between gap theorems and order Tauberian theorems.

2. Tauberian gap theorems with restrictions. We shall consider the following condition for a sequence $\{s_n\}$. Let $q_0 < q_1 < \dots$ be positive integers and let $0 < G_n \nearrow +\infty$. We assume:

- (3) (a) $u_n = 0$ if $n \neq q_j$, $j = 0, 1, \dots$,
 (b) $s_n = O(G_n)$.

We have:

Theorem 1. Let $A = (a_{nk})$ be a regular matrix method, let G_n be numbers with the properties $0 < G_n \nearrow +\infty$ and

$$(4) \quad \sum_{k=0}^{\infty} |a_{nk}| G_k < +\infty, \quad n = 0, 1, \dots$$

Then there exist indices $q_0 < q_1 < \dots$ for which (3) is a Tauberian condition for A . In addition, one can assume the q_j to belong to a given sequence of indices $n_0 < n_1 < \dots$.

Remark. The theorem is not true with (3a) only instead of (3) (see [4, p. 79]). The regularity requirement can be relaxed.

Proof. We define indices $p_n < r_n$, $n = 1, 2, \dots$, with the properties $p_n \rightarrow \infty$ and

$$(5) \quad \sum_{k=0}^{p_n-1} |a_{nk}| G_k \rightarrow 0, \quad \sum_{k=r_n+1}^{\infty} |a_{nk}| G_k \rightarrow 0, \quad n \rightarrow \infty.$$

Now we make q_j a subsequence of n_k to increase so rapidly, that each q -interval contains a (p, r) -interval. More precisely, for each j and $n = n(j)$, $[p_n, r_n] \subset [q_j, q_{j+1})$, $j = 0, 1, \dots$. Now if the sequence $\{s_n\}$ satisfies (3), then

$$\sum_{k=0}^{\infty} a_{nk} s_k = o(1) + s_{q_j} \sum_{k=p_n}^{r_n} a_{nk} + o(1), \quad n = n(j), \quad j = 0, 1, \dots$$

The regularity of A and (5) imply that the reduced row sums appearing above tend to 1. Therefore, if $\{s_n\}$ is A -summable, then $\{s_{q_j}\}$ and hence $\{s_n\}$ is convergent.

3. The main result.

Theorem 2. Let $A = (a_{nk})$ be a regular matrix method for which

$$(6) \quad \lim_{k \rightarrow \infty} \left(\sup_{n=0,1,\dots} |a_{nk}| \right) = 0;$$

let $n_0 < n_1 < \dots$ be a sequence of indices. Then there exist constants $d_n \geq 0$ with the properties $d_n \rightarrow 0$ and $\sum_0^\infty d_n = +\infty$, for which $u_n = o(d_n)$, but not $u_n = O(d_n)$ is a Tauberian condition for the method A .

Remark. Condition (6), satisfied for most methods, is not superfluous: consider $A = \text{identity}$.

Proof. We state a corollary of the condition (6). There exists a function $0 < \omega(n) \nearrow +\infty$ (summability function of the first kind) with the following property. Each bounded sequence $\{s_n\}$, for which the number of $s_k \neq 0$, $k = 0, \dots, n$, does not exceed $\omega(n)$ for $n = 1, 2, \dots$, is A -summable to zero [2, p. 181].

We take q_j, G_n from Theorem 1, and replace q_j by a subsequence (also denoted by q_j) which increases so rapidly that the following conditions are satisfied:

$$(A) \quad q_{j+1} > q_j + 2j, \quad j = 0, 1, \dots;$$

(B) the sequence $q_0, q_1, q_1 + 1, q_1 + 2, \dots, q_j, q_j + 1, \dots, q_j + 2j, \dots$ has counting function $\leq \omega(n)$;

$$(C) \quad (3a) \text{ and } u_n = O(1) \text{ imply } (3b).$$

We define $d_n = 1/j$ if $q_j \leq n \leq q_j + 2j$, $j = 0, 1, \dots$, $d_n = 0$ for all other n . The d_n have the required properties. We show that (2) is not a Tauberian condition for A , but (1) is.

Indeed, the sequence defined by $s_{q_j} = 0$, $s_{q_j+1} = 1/j, \dots, s_{q_j+2j} = 1, \dots$, $s_{q_j+2j} = 0$, $j = 0, 1, \dots$, $s_n = 0$ for other n , is A -summable to zero, and satisfies (2), yet it is divergent. On the other hand, if a sequence $\{s_n\}$ satisfies (1), then its oscillation in the interval $[q_j, q_j + 2j]$ is $o(1)$. If we write $s'_n = s_{q_j+2j}$ for $q_j \leq n \leq q_j + 2j$, $s'_n = 0$ for all other n , then $s_n = s'_n + o(1)$. The sequence $\{s'_n\}$ satisfies (3b) by (C), and it also satisfies (3a). If $\{s_n\}$ is A -summable, so is $\{s'_n\}$, and by Theorem 1, $\{s'_n\}$ is convergent. Thus also the original sequence $\{s_n\}$ is convergent.

BIBLIOGRAPHY

1. G. G. Lorentz, *Tauberian theorems and Tauberian conditions*, Trans. Amer. Math. Soc. **63** (1948), 226–234. MR **9**, 425.
2. ———, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR **10**, 367.
3. M. Stieglitz, *Die allgemeine Form der O-Tauber-Bedingung für das Euler-Knopp- und Borel-Verfahren*, J. Reine Angew. Math. **246** (1971), 172–179. MR **42** #8132.
4. K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*, 2nd ed., Ergebnisse der Math. und ihrer Grenzgebiete, Band 15, Springer-Verlag, Berlin and New York, 1970. MR **41** #8863.

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