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ON THE EXISTENCE OF POINT COUNTABLE BASES IN MOORE SPACES

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ABSTRACT. In this paper, the author answers in the negative two questions raised by E. E. Grace and R. W. Heath concerning the existence of point countable bases in Moore spaces. These answers are obtained by a general construction technique developed by the author which associates to each first countable T_2-space a Moore space.

In [4], F. B. Jones showed that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Moore space is metrizable. Hence, since each separable Moore space with a point countable base is known to be metrizable ([8] and [6]), the following question has often been raised: If $2^{\aleph_0} < 2^{\aleph_1}$, must each normal Moore space have a point countable base? In [1], E. E. Grace and R. W. Heath considered metrizability of metacompact (pointwise paracompact) Moore spaces and of Moore spaces with a point countable base. Also in [1], they raised the following two questions whose solutions would be beneficial to a solution of the above question: (1) A Moore space S is said to have property P provided that for each separable subset M of S and each open covering H of S, there exists an open covering K of S such that K refines H, and M intersects only countably many members of K. Grace and Heath noted that if $2^{\aleph_0} < 2^{\aleph_1}$, then each normal Moore space has property P.

Question 1 Does each Moore space with property P have a point countable base?

(2) *Question* 2. Is each Moore space with a point countable base metacompact?

In [7], the author described a technique which associates to each first countable T_2-space X_0 a Moore space X. In this paper it is shown that (i) if X_0 is the space of countable ordinals with the order topology, then the associated space X provides a negative answer to Question 1, and (ii) if X_0 is a certain space given in [2], then the associated space X provides a nega-

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tive answer to Question 2. Note that a negative answer to Question 2 was announced by Heath in [3], although to the author's knowledge such an example has never appeared in print. However, Professor Heath has shown his example to the author, and the author gratefully acknowledges that his solution to Question 2 was obtained with hindsight of Heath's example.

A development for a space S is a sequence $G_1, G_2, \cdots$ of open coverings of S such that (1) $G_{i+1} \subset G_i$ for each i, and (2) if $p \in S$ and D is an open set containing p then there exists an n such that each element of G_n containing p is contained in D. A Moore space is a regular T_1-space which has a development. The statement that a collection H of point sets is point finite (point countable) means that no point belongs to infinitely (uncountably) many elements of H. A space S is metacompact provided for each open covering G of S there exists a point finite open covering H of S which refines G and covers S. It follows immediately that each metacompact Moore space has a point countable base.

Theorem 1. *There exists a Moore space X such that for each open covering H of X and each separable subset M of X there exists an open covering K of X such that K refines H, and M intersects only countably many members of K but X does not have a point countable base.*

Proof. Denote by X_0 the space of countable ordinals with the order topology. For each $x \in X_0$ such that x is a limit ordinal, denote by $x_1, x_2, \cdots$ a sequence of elements of X_0 which converges to x. Then, for each i, let $u_i(x) = \{y \in X_0 | x_i$ precedes y and either y is x or y precedes $x\}$. For each $x \in X_0$ such that x is not a limit ordinal in X_0 and for each i, let $u_i(x) = \{x\}$. Note that for each $x \in X_0$, $u_1(x), u_2(x), \cdots$ forms a local base for x in X_0. Now, denote by S_1 a copy of X_0 and for each positive integer i, denote by $S_{(1,i)}$ a unique copy of X_0 distinct from S_1. Let $X = S_1 \cup (\bigcup_{i=1}^{\infty} S_{(1,i)})$ and for each $p \in X$, denote by x_p the element of X_0 which is identified with p. If $p \in X$ and j is a positive integer, define $g_j(p)$ as follows: (1) If $p \in S_{(1,i)}$ for some i, let $g_j(p) = \{p\}$. (2) If $p \in S_1$, let $g_j(p) = \{p\} \cup \{q \in S_{(1,i)} | i \geq j$ and $x_q \in u_i(x_p)$ in $X_0\}$. It follows from [7] that if $G_i = \{g_j(p) | p \in X$ and $j \geq i\}$ for each i, then $G_1, G_2, \cdots$ is a development for the nonnormal Moore space X. To see that X is not normal consider $H = \{p \in S_1 | x_p$ is a limit ordinal in $X_0\}$ and $K = \{p \in S_1 | x_p$ is not a limit ordinal in $X_0\}$. It is easily seen that H and K are two mutually exclusive closed sets in X which cannot be separated by mutually exclusive open sets.

Claim 1. X is locally separable. To see this, for each $p \in X$ and each

j, consider $g_j(p)$. Note that since each initial segment in X_0 is countable, by definition so is $g_j(p)$.

Claim 2. Each separable subset M of X is countable. For if M is separable, $M \cap S_{(1,i)}$ is countable for each i, since each point of X not on S_1 is isolated. Thus, $T = M \cap (\bigcup_{i=1}^{\infty} S_{(1,i)})$ is countable. Now, let $t \in S_1$ such that if $p \in T$ then x_p is x_t or x_p precedes x_t in X_0. It follows that if $q \in S_1$ and x_q follows x_t in X_0, then q is not in the closure of T in X. Thus, since no point of S_1 is a limit point of S_1, there are at most countably many points q of $S_1 \cap M$ such that x_q follows x_t in X_0. Hence, M is countable.

Claim 3. If M is a separable subset of X and H is an open covering of X, then there exists an open covering K of X such that K refines H, and M intersects only countably many elements of K. By Claim 2, M is countable. Thus, let $t \in S_1$ such that if $p \in M$ then either x_p is x_t or x_p precedes x_t in X_0. Now, if $q \in X$ such that x_q does not follow x_t in X_0, let $g(q)$ be an open set containing q which is contained in an element of H. If $q \in X$ such that x_q follows x_t, let $g(q)$ be an open set containing q which is contained in an element of H and which does not intersect $\{p \in X | x_p$ precedes x_t or x_p is $x_t\}$. If $K = \{g(q) | q \in X\}$, then K has the desired properties.

Claim 4. X does not have a point countable base. By the proof of [8, Theorem 1], each locally separable Moore space with a point countable base is metrizable. This completes the proof.

Theorem 2. *There exists a Moore space with a point countable base that is not metacompact.*

Proof. Denote by X_0 a well-ordered uncountable subset of the x-axis such that each initial segment is countable. Let T_1 denote the usual topology on X_0 and let T_2 denote the collection of final segments of X_0. Now, let T denote the supremum of T_1 and T_2. The space (X_0, T) was given in [2] as an example of a hereditarily Lindelöf, first countable T_2-space which is not separable. For each point $x \in X_0$ and each i, let $u_i(x) = \{y \in X_0 | y \in (x - 1/i, x + 1/i)$ in T_1 and either x is y or x precedes y in the well-ordering of $X_0\}$. Then for each $x \in X_0$, $u_1(x), u_2(x), \cdots$ is a local base for x in (X_0, T). As in the proof of Theorem 1, denote by S_1 a copy of X_0 and for each positive integer i, denote by $S_{(1,i)}$ a unique copy of X_0 distinct from S_1. Let $X = S_1 \cup (\bigcup_{i=1}^{\infty} S_{(1,i)})$ and for each $p \in X$, denote by x_p the element of X_0 which is identified with p. If $p \in X$ and j is a positive integer, define $g_j(p)$ as follows: (1) If $p \in S_{(1,i)}$ for some i, let $g_j(p) = \{p\}$. (2) If $p \in S_1$,

let $g_j(p) = \{p\} \cup \{q \in S_{(1,i)} | i \geq j$ and $x_q \in u_i(x_p)$ in $(X_0, T)\}$. Again, it follows that if $G_i = \{g_j(p) | p \in X$ and $j \geq i\}$ for each i, then $G_1, G_2, \cdots$ is a development for the Moore space X.

Claim 1. X has a point countable base. Observe that for each $p \in X$ and each i, if $p \in g_i(q)$ for some $q \in X$, then x_q is x_p or x_q precedes x_p in the well-ordering of X_0. Hence, each point p of X is contained in at most countably many elements of G_1.

Claim 2. X is not metacompact. For suppose that H is a point finite open covering of X which refines G_1. Note that each element of H contains at most one point of S_1, and since S_1 is uncountable, there exists a positive integer n such that $K = \{p \in S_1 | g_n(p)$ is contained in an element of $H\}$ is uncountable. Consider $M = \{x_q \in X_0 | q \in K\}$. Since M is uncountable, by [5, Chapter 1, Theorem 6], there exists a point $x \in M$ such that x is a limit point with respect to T_1 of $\{x_q \in M | x_q$ precedes x in the well-ordering of $X_0\}$. But, it follows that there exists a point $r \in S_{(1,i)}$ such that $x_r = x$ and r is contained in infinitely many elements of H. This completes the proof.

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