APPROXIMATION BY CONTINUED FRACTIONS

MELVYN B. NATHANSON

ABSTRACT. Let x be a real irrational number whose continued fraction has infinitely many partial quotients not less than k . A simple proof is given that the inequality $|x - p/q| < (k^2 + 4)^{-1/2} q^{-2}$ has infinitely many rational solutions p/q . The constant $(k^2 + 4)^{-1/2}$ is best possible.

Recently, J. H. E. Cohn [1] gave a simple proof of Hurwitz' theorem, that for any real irrational number x there are infinitely many rational numbers p/q such that $|x - p/q| < 5^{-1/2} q^{-2}$. In this note we make a small change in Cohn's proof and get a much stronger result.

For $k \geq 1$, let $F(k)$ be the set of all real numbers x such that $0 \leq x \leq 1$ and the continued fraction for x has no partial quotient greater than k . Let $F(0) = \emptyset$. Real numbers x and y are called equivalent if their continued fractions eventually coincide.

Theorem. Let $k \geq 1$ and let x be a real irrational number not equivalent to an element of $F(k-1)$. Then there are infinitely many rational numbers p/q such that

$$(1) \quad |x - p/q| < 1/(k^2 + 4)^{1/2} q^2.$$

The constant $(k^2 + 4)^{-1/2}$ is best possible.

Proof. Let p_n/q_n denote the n th convergent of the continued fraction $[a_0, a_1, a_2, \dots]$ of x . Following Cohn, we let $\theta_n = q_n |q_n x - p_n|$ and $\phi_n = \min(\theta_{n-1}, \theta_n, \theta_{n+1})$ and obtain

$$\frac{\theta_n}{q_n^2} + \frac{\theta_{n+1}}{q_{n+1}^2} = \left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}},$$

and so

$$(2) \quad \frac{q_{n+1}}{q_n} = \frac{1 \pm (1 - 4\theta_n \theta_{n+1})^{1/2}}{2\theta_n}.$$

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Then

$$\frac{1 + (1 - 4\theta_n \theta_{n+1})^{1/2}}{2\theta_n} \geq \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} \geq a_{n+1} + \frac{2\theta_{n-1}}{1 + (1 - 4\theta_{n-1} \theta_n)^{1/2}},$$

and

$$2\phi_n a_{n+1} \leq 2\theta_n a_{n+1} \leq (1 - 4\theta_n \theta_{n+1})^{1/2} + (1 - 4\theta_{n-1} \theta_n)^{1/2} \leq 2(1 - 4\phi_n^2)^{1/2},$$

or $\phi_n \leq (a_{n+1}^2 + 4)^{-1/2}$. If $\phi_n = (a_{n+1}^2 + 4)^{-1/2}$, then $\theta_n = \theta_{n+1} = \phi_n$. But, by (2), this is impossible. Therefore, $\phi_n < (a_{n+1}^2 + 4)^{-1/2}$ for all n .

If x is not equivalent to an element of $F(k-1)$, then for infinitely many n we have $a_{n+1} \geq k$, and so $\phi_n < (k^2 + 4)^{-1/2}$. This means that infinitely many convergents of the continued fraction for x satisfy (1).

The continued fraction for $x = ((k^2 + 4)^{1/2} - k)/2$ is $[0, k, k, k, \dots]$, and so x is not equivalent to an element of $F(k-1)$. If $A < (k^2 + 4)^{-1/2}$, it follows by a standard argument (see [2, Theorem 194] in the case $k = 1$) that $|x - p/q| < Aq^{-2}$ has only finitely many rational solutions p/q , and so the constant $(k^2 + 4)^{-1/2}$ is best possible.

Remark. For $k = 1$, this theorem gives Hurwitz' theorem. The set $F(1)$ contains only the number $(5^{1/2} - 1)/2$, the golden mean, and so when $k = 2$ the theorem gives the well-known result that, for any irrational number x not equivalent to the golden mean, the inequality $|x - p/q| < 8^{-1/2} q^{-2}$ has infinitely many rational solutions p/q .

REFERENCES

1. J. H. E. Cohn, *Hurwitz' theorem*, Proc. Amer. Math. Soc. 38 (1973), 436.
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBON-DALE, ILLINOIS 62901

Current address: School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540