REPRESENTATION THEOREMS FOR UNCONDITIONALLY CONVERGING OPERATORS

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ABSTRACT. Let $N(X)$ be the set $\{F \in X'' : \text{there exists a weakly unconditionally converging series } \sum x_n \text{ in } X \text{ such that } F = \sigma(X'', X') - \lim_n \sum_{i=1}^n Jx_i\}$. Representation theorems for the unconditionally converging operators (map weakly unconditionally converging series into unconditionally converging series) are developed by using the $\sigma(X', N(X))$ topology of X' .

Let T be a continuous linear operator which maps a Banach space X into a Banach space Y . T is said to be unconditionally converging (uc) if T maps weakly unconditionally converging (wuc) series into unconditionally converging (uc) series. A series $\sum x_i$ in X is uc if for each subseries $\sum x_{k_i}$, there is an element $x \in X$ such that $x = \sigma(X, X') - \lim_n \sum_{i=1}^n x_{k_i}$; and is wuc if $\sum |f(x_i)| < \infty$ for every $f \in X'$. $N(X)$ is to denote the set $\{F \in X'' : \text{there exists a wuc series } \sum x_n \text{ in } X \text{ such that } F = \sigma(X'', X') - \lim_n \sum_{i=1}^n Jx_i\}$. It is clear that $N(X)$ is a linear subspace of X'' containing JX (J is the natural map of X into X''). Note that $\sigma(X', N(X))$ is a topology on X' weaker than (or equal to) the weak topology and stronger than (or equal to) the weak* topology. A subset B' of X' is said to be wuc-limited if $\lim_n \sup_{B'} x'(x_n) = 0$ for every wuc series $\sum x_n$ in X .

Operator representation theorems are given for compact and weakly compact operators in [2] and [6]. We develop similar representation theorems for the unconditionally converging operators by using the $\sigma(X', N(X))$ topology.

1. **Characterizing a uc operator.** Let T' and T'' denote respectively the first and second adjoint of T . The uc operator can be characterized as follows.

Theorem 1.1. *Let $T: X \rightarrow Y$. The following are equivalent.*

(a) T is uc.

Received by the editors October 12, 1972.

AMS (MOS) subject classifications (1970). Primary 46G10.

Key words and phrases. Operator representations, unconditionally converging operator, unconditionally converging series.

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(b) T' maps bounded sets of Y' into wuc-limited sets of X' .

Proof. (a) $\Rightarrow$ (b): Let T be uc and Σx_n be wuc. Then ΣTx_n is uc. Hence, $Tx_n \rightarrow 0$, and we obtain

$$0 = \lim_n \sup_{B'} y'(Tx_n) = \lim_n \sup_{T'B'} T'y'(x_n).$$

Therefore, $T'B'$ is wuc-limited.

(b) $\Rightarrow$ (a): Now suppose Σx_n is wuc. Let U' be the unit disk of X' . Then

$$0 = \lim_n \sup_{T'U'} T'y'(x_n) = \lim_n \sup_{U'} y'(Tx_n) = \lim_n \|Tx_n\|.$$

Hence, $\inf_n \|Tx_n\| = 0$, so by [1, Theorem 5], ΣTx_n is uc.

Corollary 1.2. Let B' be a bounded set in X' . Then B' is $\sigma(X', N(X))$ -conditionally compact if and only if B' is a wuc-limited set.

Proof. Define $S = \{x \in X: |b'(x)| \leq 1 \text{ for every } b' \in B'\}$. Then S is a closed subset in X by [7, p. 238]. If p is the gauge of S , then p' is a norm on $X/\text{Ker } p$, where $p'(\hat{x}) = p(x)$, $x = \hat{x} + \text{Ker } p$. Let T be the natural map from X to $Y = X/\text{Ker } p$. If U' is the unit disk of Y' , then $T'U' = B'$. Now B' is $\sigma(X', N(X))$ conditionally compact $\Leftrightarrow T$ is uc $\Leftrightarrow T'$ maps U' onto a wuc-limited set $\Leftrightarrow B'$ is a wuc-limited set.

2. Representation theorems for uc operators. Let S be a compact Hausdorff space and let T be a continuous linear operator from X to $C(S)$. From [2, p. 490] we know there exists a mapping $v: S \rightarrow X'$ which is continuous with the $\sigma(X', X)$ topology such that:

(a) $Tx(s) = v(s)x$ for $x \in X$ and $s \in S$;

(b) $\|T\| = \sup_{s \in S} |v(s)|$.

Conversely, if given such a map v , then the operator T defined by (a) is a continuous linear operator from X to $C(S)$ with norm given by (b). Here v is defined by $v = T'\pi$ where $\pi: S \rightarrow C(S)$ is given by $\pi(s)f = f(s)$, $f \in C(S)$, $s \in S$. π is a homeomorphism of S into a compact subset of $C'(S)$ with the $\sigma(C'(S), C(S))$ topology.

Theorem 2.1. T is uc if and only if v is $\sigma(X', N(X))$ continuous.

Proof. If T is uc, then T' is $\sigma(C'(S), C(S)) - \sigma(X', N(X))$ continuous; thus $v = T'\pi$ will be $\sigma(X', N(X))$ continuous.

Conversely, suppose v is $\sigma(X', N(X))$ continuous. Let Σy_n be a wuc series in X and Σx_n be an arbitrary subseries of Σy_n . It suffices to show that $B = \{\Sigma_{i=1}^n Tx_i | n = 1, 2, \dots\}$ is $\sigma(C(S), C'(S))$ conditionally compact,

since then T would be uc (i.e. $\sum T y_n$ would be $\sigma(C(S), C'(S))$ subseries convergent, hence uc). $B \subseteq C(S)$, so by [2, Theorem 14, p. 269] B is $\sigma(C(S), C'(S))$ conditionally compact if and only if B is bounded and quasi-equitcontinuous. But $B = TA$ where $A = \{\sum_{i=1}^n x_i \mid n = 1, 2, \dots\}$, and A is bounded, so B is bounded.

To see that B is quasi-equitcontinuous, let $s_\alpha \rightarrow s$ in S , let $\epsilon > 0$, and let α_0 be given. Since v is $\sigma(X', N(X))$ continuous, $v(s_\alpha) \rightarrow v(s)$ in $\sigma(X', N(X))$. Since $\sum x_n$ is wuc, $\sum_{i=1}^n Jx_i \xrightarrow{w^*} F \in N(X)$. Thus, $A_1 = A \cup \{F\}$ is a $\sigma(X'', X')$ compact set and $v(s_\alpha)$ and $v(s)$ are in $C(A_1)$. Since $v(s) \in C(A_1)$, then by Arzela's theorem [2, Theorem 11, p. 268] the convergence $v(s_\alpha) \rightarrow v(s)$ is quasi-uniform on A_1 , hence, quasi-uniform on A . Thus, there exists a finite set of indices $\alpha_1, \dots, \alpha_n \geq \alpha_0$ such that for each $a \in A$

$$\begin{aligned} \min_{1 \leq i \leq n} |v(s_{\alpha_i})a - v(s)a| &< \epsilon; \\ \min_{1 \leq i \leq n} \left| v(s_{\alpha_i}) \sum_{j=1}^k x_j - v(s) \sum_{j=1}^k x_j \right| &< \epsilon \quad \text{for each } k; \\ \min_{1 \leq i \leq n} \left| \sum_{j=1}^k T x_j(s_{\alpha_i}) - \sum_{j=1}^k T x_j(s) \right| &< \epsilon \quad \text{for each } k; \\ \min_{1 \leq i \leq n} |f(s_{\alpha_i}) - f(s)| &< \epsilon \quad \text{for each } f \in B. \end{aligned}$$

Thus, B is quasi-equitcontinuous, and it therefore follows that B is $\sigma(C(S), C'(S))$ conditionally compact, and hence T is uc.

Let S be an arbitrary topological space and $B(S)$ the space of bounded scalar-valued functions on S with the sup norm. Then the operator representation theorem is appreciably simpler to prove.

Theorem 2.2. *Let $T: X \rightarrow B(S)$ and $p: S \rightarrow X'$ be such that p is a bounded map with $Tx(s) = p(s)x$. Then T is uc if and only if $p(S)$ is $\sigma(X', N(X))$ conditionally compact.*

Proof. By Corollary 1.2, it suffices to prove the theorem for $p(S)$ a wuc-limited set.

Assume T is uc and let $\sum x_n$ be a wuc series. Then $\sum T x_n$ is a uc series; hence, $T x_n \rightarrow 0$, so

$$\lim_n \sup_{p(S)} p(s)x = \lim_n \sup_S T x_n(s) = 0.$$

Therefore, $p(S)$ is a wuc-limited set.

Conversely, let $p(S)$ be any bounded subset of X' and assume $p(S)$ is wuc-limited. Let $\sum x_n$ be a wuc series. Then

$$\lim_n \sup_S Tx_n(s) = \lim_n \sup_{p(S)} p(s)x_n = 0.$$

Hence, $\inf_n \|Tx_n\| = 0$, so by [1, Theorem 5], $\sum Tx_n$ is uc.

Denote the bounded operators from X to Y by $B(X, Y)$ and the uc operators by $UC(X, Y)$.

Corollary 2.3. *The following conditions are equivalent.*

- (a) $B(X, B(S)) = UC(X, B(S))$.
- (b) U' (the unit disk of X') is $\sigma(X', N(X))$ compact.
- (c) U' is wuc-limited.

Proof. This follows from Theorem 2.2 and Corollary 1.2.

REFERENCES

1. C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, *Studia Math.* **17** (1958), 151–164. MR **22** #5872.
2. N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, *Pure and Appl. Math.*, vol. 7, Interscience, New York, 1958. MR **22** #8302.
3. J. Howard, *The comparison of an unconditionally converging operator*, *Studia Math.* **33** (1969), 295–298. MR **40** #785.
4. J. Howard and K. Melendez, *Characterizing operators by their first and second adjoints* (to appear).
5. C. W. McArthur, *A note on subseries convergence*, *Proc. Amer. Math. Soc.* **12** (1961), 540–545. MR **23** #A3443.
6. E. D. Thorp and R. J. Whitley, *Operator representation theorems*, *Illinois J. Math.* **9** (1965), 595–601. MR **31** #6126.
7. A. Wilansky, *Functional analysis*, Blaisdell, New York, 1964. MR **30** #425.

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