ITERATED LITTLEWOOD-PALEY FUNCTIONS AND A MULTIPLIER THEOREM

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ABSTRACT. A sufficient condition for a bounded function to be a multiplier of Fourier transforms on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is established. The classical case of Marcinkiewicz is properly included. The main tools used in obtaining this result are iterated variants of the classical Littlewood-Paley functions together with an L^p estimate on certain maximal functions closely related to strong differentiability of multiple integrals.

Introduction. The classical version of the Marcinkiewicz multiplier theorem for Fourier integrals (also known as Mihlin's or Lizorkin's theorem) states that if $m(\xi)$ is a function on $\mathbb{R}^n$ such that $|\xi^\alpha D^\alpha m(\xi)|$ are bounded for all α 's in $\mathcal{Q}$, where $\mathcal{Q}$ is the collection of all multi-indexes whose components are zero or one, then the transformation $f \rightarrow Mf$, defined on C^∞ functions with compact support by $(Mf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$, maps $L^p(\mathbb{R}^n)$ boundedly into itself (see [4], [6]). The purpose of this note is to show that the theorem remains true if the set $\mathcal{Q}$ is replaced by a more general collection of multi-indexes. The main tools used in obtaining our result are iterated variants of the classical Littlewood-Paley functions, g and g_λ^* , together with an estimate on certain maximal functions closely related to strong differentiability of multiple integrals.

We use the following notation: $\mathbb{R}^n$ is a n -dimensional real Euclidean space whose elements are denoted by x, z and ξ . Variables in $\mathbb{R}^1$ are denoted by y and s . T^n represents the cone $\{t = (t_1, \dots, t_n): t_i > 0, i = 1, \dots, n\}$ and $\Pi t = \prod_{i=1}^n t_i$. For $x \in \mathbb{R}^n$ and $t \in T^n$, tx denotes the point $(t_1 x_1, \dots, t_n x_n)$ in $\mathbb{R}^n$. If Ω is a measurable subset of $\mathbb{R}^n$, $|\Omega|$ denotes the Lebesgue measure of Ω and $t\Omega = \{tx: x \in \Omega\}$. For x and y in $\mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ and $|x| = \langle x, x \rangle^{1/2}$. All integrals are with respect to Lebesgue measure, dx , and over all of $\mathbb{R}^n$, unless denoted otherwise. $L^p = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is the usual Banach space of measurable complex valued

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functions f , such that $\|f\|_p = (\int |f(x)|^p dx)^{1/p} < \infty$; L^∞ is the space of essentially bounded functions.

For f in L^1 , the Fourier transform, $f \rightarrow \hat{f}$, is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-i(x, \xi)} dx.$$

The Fourier transform is defined on the space of tempered distributions in the usual manner. All Fourier transforms and differentiations are to be interpreted in the distribution sense, unless they make sense otherwise. The convolution of two reasonable functions, f_1 and f_2 , is defined by

$$f_1 * f_2(x) = \int f_1(x-z) f_2(z) dz,$$

whenever this operation makes sense. If f is in L^1 and $t \in T^n$ then f_t is the function defined by $f_t(x) = (\Pi t)^{-1} f(t^{-1}x)$, where $t^{-1} = (t^{-1}, \dots, t_n^{-1}) \in T^n$.

Multi-indexes always have nonnegative integer components and are denoted by the symbols α, β , and γ . If x is in R^n , $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ and if f is a distribution $D^\alpha f = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n} f$. $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$, $i = 1, \dots, n$.

The symbol C will be used generically for constants appearing in certain estimates. It need not be the same at different occurrences.

1. The Littlewood-Paley functions. A measurable complex valued function ϕ defined on R^1 is said to satisfy condition A if there are positive constants C and δ such that

- (i) $|\phi(y)| \leq C(1 + |y|)^{-1-\delta}$,
- (ii) $\int_{-\infty}^{\infty} \phi(y) dy = 0$, and
- (iii) $\int_{-\infty}^{\infty} |\phi(s-y) - \phi(s)| ds \leq C|y|^\delta$.

Suppose that $k_1, \dots, k_n$ satisfy condition A and set $K(x) = \prod_{i=1}^n k_i(x_i)$, $x = (x_1, \dots, x_n)$. For f in L^p we define the Littlewood-Paley function $G_K(f)$ by the formula

$$G_K(f, x) = \left\{ \int_{T^n} |K_t * f(x)|^2 \frac{dt}{\Pi t} \right\}^{1/2}$$

Theorem 1. *If f is in L^p , $1 < p < \infty$, then $G_K(f)$ is in L^p and $\|G_K(f)\|_p \leq C\|f\|_p$ where C is a constant which depends only on K and p .*

The theorem follows by iterating the vector-valued analogue of the

Calderón-Zygmund inequality. To see this, write

$$K_t * f(x) = \int_{-\infty}^{\infty} \frac{1}{t_n} k_n\left(\frac{z_n}{t_n}\right) \cdots \left(\int_{-\infty}^{\infty} \frac{1}{t_1} k_1\left(\frac{z_1}{t_1}\right) f(x-z) dz_1 \right) \cdots dz_n$$

and apply the results of [1, p. 359 and p. 363] n times.

Theorem 2. Suppose $k_1, \dots, k_n$ are even functions which satisfy condition A and set $K(x) = \prod_{i=1}^n k_i(x_i)$. If none of the k_i 's are identically zero, then for $f \in L^p$, $1 < p < \infty$, $\|f\|_p \leq C \|G_K(f)\|_p$ where C is a constant depending only on K and p .

The above fact is an immediate consequence of Plancherel's formula and Theorem 1. (Analogous calculations may be found in [6, p. 85] and [5, corollary to Theorem 7].)

Now suppose that H is a nonnegative function in L^1 and K satisfies the hypothesis of Theorem 1. If f is in L^p , the Littlewood-Paley function $G_{K,H}(f)$ is defined by the formula

$$G_{K,H}(f, x) = \left\{ \int_{T^n} \int H_t(x-z) |K_t * f(z)|^2 dz \frac{dt}{\prod t} \right\}^{1/2}.$$

Observe that if f is in L^2 then it follows from Plancherel's formula that $G_{K,H}(f)$ is in L^2 and $\|G_{K,H}(f)\|_2 \leq C \|f\|_2$ where

$$C = \left\{ \|H\|_1 \left\| \int_{T^n} |\hat{K}_t(\xi)|^2 \frac{dt}{\prod t} \right\|_{\infty} \right\}^{1/2}.$$

Using arguments similar to those found in [6, p. 91] and [5, Theorem 8], we see the following.

Theorem 3. Suppose the sublinear transformation $f \rightarrow (Hf)^*$, defined by $(Hf)^*(x) = \sup_{t \in T^n} |H_t * f(x)|$, maps L^q boundedly into L^q for each q , $1 < q \leq \infty$. If f is in L^p , $2 \leq p < \infty$, then $G_{K,H}(f)$ is in L^p and $\|G_{K,H}(f)\|_p \leq C \|f\|_p$, where C depends only on K , H , and p .

2. Tent functions. In order to apply the results of the previous section, we need a class of H 's which satisfy the hypothesis of Theorem 3. We begin by introducing an ordering in each 2^n -ant of R^n .

Write $\text{sgn } x = (\text{sgn } x_1, \dots, \text{sgn } x_n)$. If $\text{sgn } x = \text{sgn } z$ and $|x_i| \leq |z_i|$, $i = 1, \dots, n$, we write $x \leq z$.

Definition. h is said to be a *tent function* if h is a real valued nonnegative function in L^1 and $h(x) \geq h(z)$ whenever $\operatorname{sgn} x = \operatorname{sgn} z$ and $x \leq z$.

Theorem 4. Suppose H is in L^1 and $|H(x)| \leq h(x)$, almost everywhere, where h is a tent function. If f is in L^p , $1 < p \leq \infty$, then $(Hf)^*(x) = \sup_{t \in T^n} |H_t * f(x)|$ is in L^p and $\|(Hf)^*\|_p \leq C \|f\|_p$ where C depends only on h , p , and n .

The result is an immediate consequence of the following lemmas. For any locally integrable f , define

$$f^*(x) = \sup_{t \in T^n} \frac{1}{|tQ|} \int_{tQ} |f(x-z)| dz$$

where Q is the cube $\{x: -1 \leq |x_i| \leq 1\}$.

Lemma 1. (Jessen, Marcinkiewicz, and Zygmund). If $f \in L^p$, $1 < p \leq \infty$, then f^* is in L^p and $\|f^*\|_p \leq C \|f\|_p$ where C is a constant which depends only on n and p .

Lemma 2. Suppose H satisfies the hypothesis of Theorem 4. If f is in L^p , $1 < p \leq \infty$, then $\sup_{t \in T^n} |H_t * f(x)| \leq 2^{3n} \|h\|_1 f^*(x)$.

Proof. The proof of Lemma 1 may be found in [3]. To see Lemma 2, we begin by estimating the L^1 norm of h in terms of "lacunary parallelepipeds." Define $\bar{j} = (j_1, \dots, j_n)$ where the j_i 's are integers. $Q_{\bar{j}}$ denotes the parallelepiped $\{x: 2^{j_i} < x_i \leq 2^{j_i+1}, i = 1, \dots, n\}$, and $Q_{\bar{j}}^*$ denotes the parallelepiped $\{x: |x_i| \leq 2^{j_i+1}, i = 1, \dots, n\}$. Clearly $Q_{\bar{j}} \subset Q_{\bar{j}}^*$ and $|Q_{\bar{j}}| = 2^{|\bar{j}|}$, $|Q_{\bar{j}}^*| = 2^{2n} 2^{|\bar{j}|}$, where $|\bar{j}| = j_1 + \dots + j_n$. Define h_1 by $h_1(x) = h(x)$ if $\operatorname{sgn} x > (0, 0, \dots, 0)$ and 0 otherwise. Observe that

$$2^{-n} \sum_{\bar{j}} h(2^{\bar{j}}) |Q_{\bar{j}}| \leq \int h_1(x) dx \leq \sum_{\bar{j}} h(2^{\bar{j}}) |Q_{\bar{j}}|$$

where $2^{\bar{j}}$ denotes the point $(2^{j_1}, \dots, 2^{j_n})$ in R^n .

Now, if $f \in L^p$, $1 < p \leq \infty$, write

$$|H_t * f(x)| \leq \int h_t(z) |f(x-z)| dz = \sum_{i=1}^n \int (h_i)_t(z) |f(x-z)| dz,$$

where $h_i(x) = h(x)$ when x is in the i th 2^n -ant and zero otherwise, and observe that

$$\begin{aligned} \int (b_1)_t(z) |f(x-z)| dz &= \sum_{\bar{j}} \int_{t^{-1}z \in Q_{\bar{j}}} (\Pi t)^{-1} b_1(t^{-1}z) |f(x-z)| dz \\ &\leq \sum_{\bar{j}} b_1(2^{\bar{j}}) |Q_{\bar{j}}^*| \frac{1}{|tQ_{\bar{j}}^*|} \int_{tQ_{\bar{j}}^*} |f(x-z)| dz. \end{aligned}$$

Hence

$$\sup_{t \in T} \int (b_1)_t(z) |f(x-z)| dz \leq \left(\sum_{\bar{j}} b_1(2^{\bar{j}}) |Q_{\bar{j}}^*| \right) f^*(x) \leq 2^{3n} \|b_1\|_1 f^*(x).$$

An analogous argument on $b_2, \dots, b_{2n}$, gives the desired result.

3. A multiplier theorem. Suppose $\mathcal{Q}$ is a set of multi-indexes. We say that $\mathcal{Q}$ is *nice* if

- (i) $(\sum_{\alpha \in \mathcal{Q}} |x^\alpha|)^{-2}$ is in L^1 , and
- (ii) if $\alpha \in \mathcal{Q}$ then any β satisfying $\beta \leq \alpha$ is also in $\mathcal{Q}$.

The following are examples of nice sets of multi-indexes: the set mentioned in the introduction; $\mathcal{Q} = \{\alpha: |\alpha| \leq \kappa\}$, where κ is any integer greater than $n/2$; $\mathcal{Q} = \{\alpha: \alpha \leq \beta^{(i)} \text{ for some } i, i = 1, \dots, n\}$, where $\beta^{(i)}$ is the multi-index which contains b_i in the i th slot, zero elsewhere, and $\sum_{i=1}^n (b_i)^{-1} < 2$.

For $t \in T^n$, we define Q_t to be the set $\{x: t_i < |x_i| < 2t_i, i = 1, \dots, n\}$; Q_t is the union of 2^n parallelepipeds.

Theorem 5. Suppose m is in L^∞ and $\sum_{\alpha \in \mathcal{Q}} (1/|Q_t|) \int_{Q_t} |\xi^\alpha D^\alpha m(\xi)|^2 d\xi \leq B^2$ for all $t \in T^n$, where $\mathcal{Q}$ is a nice set of multi-indexes. Define Mf for f in $L^2 \cap L^p$ by the formula $(Mf)^\wedge(\xi) = m(\xi) \hat{f}(\xi)$. Then $\|Mf\|_p \leq CB \|f\|_p$ for $1 < p < \infty$, where C is independent of f .

The proof of the theorem consists of picking appropriate K_1 , K_2 and H , establishing the inequality $G_{K_1}(Mf, x) \leq CG_{K_2, H}(f, x)$, and applying Theorems 2 and 3.

Let ϕ be an even function in $C^\infty(R^1)$, which is not identically zero and whose support is contained in $\{y: 1 \leq |y| \leq 2\}$. Define k_1 and k_2 to be the inverse Fourier transforms of $\phi(y)$ and $[\phi(y)]^2$ respectively (as functions in R^1) and set $K_1(x) = \prod_{i=1}^n k_1(x_i)$ and $K_2(x) = \prod_{i=1}^n k_2(x_i)$. Let $H(x) = (\sum_{\alpha \in \mathcal{Q}} |x^\alpha|)^{-2}$, where $\mathcal{Q}$ is the nice set of multi-indexes of Theorem 5. Clearly K_1 and K_2 satisfy the hypothesis of Theorems 1 and 2 and H is a tent function.

Lemma 3. *If m satisfies the hypothesis of Theorem 5 and K_1, K_2, H , and Mf are defined as above, then $G_{K_2}(Mf, x) \leq CBG_{K_1, H}(f, x)$ where C depends only on K_1 and $\mathcal{Q}$.*

Proof. Write

$$(K_2)_t * Mf(x) = \int K_1(z; m, t)(K_1)_t * f(x - z) dz$$

where

$$K_1(z; m, t) = (2\pi)^{-n/2} \int m(\xi) \Phi(t\xi) e^{i\langle z, \xi \rangle} d\xi,$$

and $\Phi(t\xi) = \prod_{j=1}^n \phi(t_j \xi_j)$. Now,

$$(1) \quad |(K_2)_t * Mf(x)|^2 \leq I_1 I_2$$

where

$$I_1 = \int \left| \sum_{\alpha \in \mathcal{Q}} (t^{-1}z)^\alpha K_1(z; m, t) \right|^2 dz$$

and

$$I_2 = \int (\prod_t) H_t(z) |(K_1)_t * f(x - z)|^2 dz.$$

To estimate I_1 , apply Plancherel's formula

$$\int |(t^{-1}z)^\alpha K_1(z; m, t)|^2 dz = \int |(t^{-1})^\alpha D^\alpha (m(\xi) \Phi(t\xi))|^2 d\xi,$$

and observe that

$$(t^{-1})^\alpha D^\alpha (m(\xi) \Phi(t\xi)) = \sum_{\beta + \gamma = \alpha} C_\beta (t^{-1})^\beta (D^\beta m(\xi)) D^\gamma \Phi(t\xi).$$

Since $\Phi(t\xi)$ is supported in $\mathcal{Q}_{1/t} = \{\xi: t_i^{-1} \leq |\xi_i| \leq 2t_i^{-1}, i = 1, \dots, n\}$, the last two formulas imply that

$$(2) \quad I_1 \leq C \sum_{\alpha \in \mathcal{Q}} \int_{\mathcal{Q}_{1/t}} |\xi^\alpha D^\alpha m(\xi)|^2 d\xi \leq CB^2 |\mathcal{Q}_{1/t}|.$$

Substituting (2) and the formula for I_2 into (1) and integrating over T^n gives us

$$\int_{T^n} |(K_2)_t * Mf(x)|^2 \frac{dt}{\prod t} \leq CB^2 \int_{T^n} \int H_t(z) |(K_1)_t * f(x - z)|^2 dz \frac{dt}{\prod t},$$

which is the desired result.

Thus the theorem holds for $2 \leq p < \infty$. The fact that it remains true in the range $1 < p \leq 2$, follows by duality (see [2, p. 100]).

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