ON GENERATING FREE PRODUCTS OF LATTICES

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ABSTRACT. For a lattice K let $g(K)$ denote the cardinality of the smallest generating set of K . Let L be the free product of the lattices A and B . It is proved that $g(L) = g(A) + g(B)$. This is proved, in fact, for free products with respect to any given equational class of lattices. Some applications and generalizations are also given.

1. For a lattice L let $g(L)$ denote the smallest cardinal m such that L has a generating set of cardinality m . In this note we compute $g(L)$ for a free product.

Theorem. *Let L be the free product of A and B . Then*

$$g(L) = g(A) + g(B).$$

In fact, we prove this result for free $\mathbf{K}$ -products, that is, for free products in an arbitrary nontrivial equational class $\mathbf{K}$.

Among the consequences of this result is that if L is n -generated (n is a positive integer), then L cannot be represented as a free product of more than n lattices and, furthermore, L is a free product of n lattices iff L is the free lattice on n generators.

The theorem is proved in §2. Some consequences of the theorem are given in §3, while some generalizations are discussed in §4. We refer the reader to [1] for notation and for concepts not defined in this note.

2. Let $\mathbf{K}$ be an equational class of lattices; let $\mathbf{K}$ be nontrivial, that is, let $\mathbf{K}$ contain at least one lattice with more than one element.

Let $L \in \mathbf{K}$; let A and B be sublattices of L . L is called a free $\mathbf{K}$ -product of A and B iff $A \cup B$ generates L and for any lattice $C \in \mathbf{K}$ and lattice homomorphisms ϕ_A and ϕ_B of A and B , respectively, into C , there exists a homomorphism ϕ of L into C such that ϕ extends both ϕ_A and ϕ_B . In this section, let L be a free $\mathbf{K}$ -product of A and B .

Presented to the Society, January 16, 1974; received by the editors September 10, 1973.
 AMS (MOS) subject classifications (1970). Primary 06A20.

Key words and phrases. Lattice, free product, generating set.

¹ Research of both authors was supported by the National Research Council of Canada.

Let L^b denote L with a new 0 and 1 adjoined to L . Let ψ be the common extension of the identity map on A and of $B \rightarrow 0$; for $x \in L$ we denote by x_A the image of x under ψ and we call x_A the *lower A -cover*. Observe that $x_A \in A \cup \{0\}$; $x \geq x_A$, and if $y \in A \cup \{0\}$, $y \leq x$, then $y \leq x_A$. We define similarly x_B , and dually x^A ($\in A \cup \{1\}$) and x^B . (See [4] or [2].)

In [3] we proved the following result.

Lemma 1. *Let X be a generating set of L . Then*

$$T_A(X) = \{x_A \mid x \in X, x_A \neq 0\}$$

is a generating set of A .

By duality,

$$T^A(X) = \{x^A \mid x \in X, x^A \neq 1\}$$

is a generating set of A .

Recall, that if $x_A \in A$, then $x^B = 1$; if not, then $x^B \in B$, and $x_A \leq x^B$, a contradiction since elements of A and B are incomparable. Similarly, if $x^A \in A$, then $x_B = 0$. Finally, recall [2], [4] that if $x^B = 1$, then $x_A \in A$; analogously for all the other cases. This implies that L has a representation as a disjoint union

$$L = A \cup B \cup L_1 \cup L_2,$$

where

$$\begin{aligned} L_1 &= \{x \mid x \in L, x \notin A \cup B \text{ and } x_A \in A, x_B = 0 \text{ or } x_B \in B, x_A = 0, \\ &\quad \text{or } x^A \in A, x^B = 1 \text{ or } x^B \in B, x^A = 1\} \\ &= \{x \mid x \in L, x \notin A \cup B \text{ and } x_A, x^A \in A \text{ or } x_B, x^B \in B\}, \\ L_2 &= \{x \mid x \in L \text{ and } x_A \in A, x_B \in B \text{ or } x^A \in A, x^B \in B\}. \end{aligned}$$

The next three lemmas describe how a generating set of L can be transformed into a generating set contained in $A \cup B$ with no increase of its size.

Lemma 2. *Let X be a generating set of L and let $x \in X \cap L_1$. Then there is an $\bar{x} \in A \cup B$ such that $T_1 = (X - \{x\}) \cup \{\bar{x}\}$ is a generating set of L , and $|X| \geq |T_1|$.*

Proof. $x \in L_1$ so let, e.g., $x_A \in A$ and $x_B = 0$. We set $\bar{x} = x_A$. To

show that T_1 generates L it is sufficient to show that T_1 generates any $a \in A$ and $b \in B$. Let $a \in A$; since X generates L , $a = p(x, x_1, \dots, x_n)$, $x_1, \dots, x_n \in X$, for a suitable polynomial p . Then, since p is isotone,

$$\begin{aligned} a = p(x, x_1, \dots, x_n) &\geq p(x_A, x_1, \dots, x_n) \\ &\geq p(x_A, (x_1)_A, \dots, (x_n)_A) = a_A = a, \end{aligned}$$

and therefore $a = p(x_A, x_1, \dots, x_n)$, as claimed.

Now let $b \in B$. Then $b = q(x, x_1, \dots, x_n)$. Recall that, by assumption, $x_B = 0$ and so $x \geq x_A \geq x_B$. Since q is isotone we obtain

$$\begin{aligned} b = q(x, x_1, \dots, x_n) &\geq q(x_A, x_1, \dots, x_n) \\ &\geq q(x_B, (x_1)_B, \dots, (x_n)_B) = b_B = b, \end{aligned}$$

and thus $b = q(x_A, x_1, \dots, x_n)$, as required.

We proceed similarly in the other three cases. This completes the proof of Lemma 2.

Lemma 3. *Let X be a generating set of L satisfying $X \cap L_1 = \emptyset$. If $\{x \mid x \in X, x_A \in A, x_B \in B\} = \emptyset$, then $X \cap (A \cup B)$ is a generating set of L .*

Proof. By Lemma 1, $T_A(X) \cup T_B(X)$ is a generating set of L . Since

$$\begin{aligned} X &= (X \cap A) \cup (X \cap B) \cup \{x \mid x \in X, x^A \in A \text{ and } x^B \in B\}, \\ T_A(X) \cup T_B(X) &= (X \cap A) \cup (X \cap B) = X \cap (A \cup B), \end{aligned}$$

completing the proof.

Lemma 4. *Let X be a generating set of L satisfying $X \cap L_1 = \emptyset$. Then there is a generating set $T_2 \subseteq A \cup B$ of L satisfying $|X| \geq |T_2|$.*

Proof. Represent X in the form

$$X = (X \cap A) \cup (X \cap B) \cup X_1 \cup X_2,$$

where

$$X_1 = \{x \mid x \in X, x_A \in A, x_B \in B\},$$

$$X_2 = \{x \mid x \in X, x^A \in A, x^B \in B\}.$$

If $X_1 = \emptyset$ then, by Lemma 3, we can set $T_2 = X \cap (A \cup B)$. By duality, if $X_2 = \emptyset$ we can set $T_2 = X \cap (A \cup B)$. So assume that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, and, say, $|X_1| \leq |X_2|$. Define

$$T_2 = (X \cap A) \cup (X \cap B) \cup T_A(X_1) \cup T_B(X_1).$$

Observe that $|X_1| \geq |T_A(X_1)|$ and $|X_2| \geq |X_1| \geq |T_B(X_1)|$, hence

$$|X_1 \cup X_2| \geq |T_A(X_1) \cup T_B(X_1)|,$$

and so $|X| \geq |T_2|$. Since $T_A(X_2) = T_B(X_2) = \emptyset$, $T_A(X) \cup T_B(X) = T_2$, a reference to Lemma 1 completes the proof.

Now we are ready to prove the theorem. Since $A \cup B$ generates L , $g(A) + g(B) \geq g(L)$ is trivial. To prove the converse, let X be a generating set with $|X| = g(L)$. By applying Lemma 2 repeatedly we obtain a generating set X' satisfying $|X| \geq |X'|$ and $X' \cap L_1 = \emptyset$. Since $|X| = g(L)$, therefore $|X'| = g(L)$. Applying Lemmas 3 and 4 we obtain a generating set $X'' \subseteq A \cup B$ with $|X'| \geq |X''|$. Again, $|X''| = g(L)$. By Lemma 1, $T_A(X'') = X'' \cap A$ generates A and $T_B(X'') = X'' \cap B$ generates B , so

$$g(L) = |X''| = |X'' \cap A| + |X'' \cap B| \geq g(A) + g(B),$$

concluding the proof of the main result.

Theorem 1. *Let $\mathbf{K}$ be a nontrivial equational class of lattices and let $L \in \mathbf{K}$ be the free $\mathbf{K}$ -product of A and B . Then $g(L) = g(A) + g(B)$.*

3. The most immediate consequence of Theorem 1 is the following.

Corollary 2. *Let $\mathbf{K}$ be a nontrivial equational class of lattices and let $L \in \mathbf{K}$ be the free $\mathbf{K}$ -product of the lattices L_i , $i \in I$. Then*

$$g(L) = \sum (g(L_i) \mid i \in I).$$

Proof. If I is finite, then this follows by induction on $|I|$ from Theorem 1. If I is infinite, then $g(L) = |L|$ and the conclusion is obvious.

Corollary 3. *Let $\mathbf{K}$ be a nontrivial equational class of lattices. Let n be a positive integer and let $L \in \mathbf{K}$ be n -generated. If L is a free $\mathbf{K}$ -product of L_i , $i \in I$, then $|I| \leq n$.*

Proof. Indeed, by Corollary 2, $n \geq g(L) = \sum (g(L_i) \mid i \in I) \geq |I|$, where the last inequality is obtained from $g(L_i) \geq 1$.

Corollary 4. *Under the conditions of Corollary 3, L is a free $\mathbf{K}$ -product of exactly n lattices iff L is $F_{\mathbf{K}}(n)$, that is, a free $\mathbf{K}$ -lattice on n generators.*

Proof. If $L = F_{\mathbf{K}}(n)$, then L is a free $\mathbf{K}$ -product of n copies of $F_{\mathbf{K}}(1)$.

Conversely, if L is n -generated and L is a free $\mathbf{K}$ -product of $L_1, \dots, L_n$, then $n \geq g(L_1) + \dots + g(L_n)$ by Corollary 3 and so $g(L_1) = \dots = g(L_n) = 1$, that is, $L_1 \cong \dots \cong L_n \cong F_{\mathbf{K}}(1)$, and $L \cong F_{\mathbf{K}}(n)$, as claimed.

4. There is a lattice situation to which our results apply that is more general than free $\mathbf{K}$ -products.

Definition 5. The lattice L is a *pseudo-free product* of its sublattices L_i , $i \in I$, iff L is generated by $\bigcup (L_i | i \in I)$ and for each $i \in I$ there are homomorphisms

$$x \rightarrow x_{L_i}, \quad L \rightarrow L_i \cup \{0\},$$

$$x \rightarrow x^{L_i}, \quad L \rightarrow L_i \cup \{1\}$$

which satisfy

$$a = a_{L_i} = a^{L_i} \quad \text{for all } a \in L_i,$$

$$a_{L_i} = 0, \quad a^{L_i} = 1 \quad \text{for all } a \in L_j, \quad j \neq i, \quad j \in I.$$

The simplest example of a pseudo-free product which is not a free $\mathbf{K}$ -product is given in Figure 1; it is a pseudo-free product of the singleton $\{a\}$ with the two-element chain $\{b_1, b_2\}$.

It is easy to check that for $a \in L_i$, $b \in L_j$, $i \neq j$, $i, j \in I$, a and b are incomparable; indeed, if $a \leq b$, then $a = a_{L_i} \leq b_{L_i} = 0$, a contradiction. Similarly, we can verify all the properties of free $\mathbf{K}$ -products that were used in proving Theorem 1 of this note. In fact, the common refinement property for free $\mathbf{K}$ -products proved in [3] relies only on these properties. We obtain:

Theorem 6. Let the lattice L be a pseudo-free product of $(A_i | i \in I)$ and of $(B_j | j \in J)$. Then L is a pseudo-free product of $(A_i \cap B_j | i \in I, j \in J, A_i \cap B_j \neq \emptyset)$, and, for $i \in I$, A_i is a pseudo-free product of $(A_i \cap B_j | j \in J, A_i \cap B_j \neq \emptyset)$.

Theorem 7. Let the lattice L be a pseudo-free product of A and B . Then $g(L) = g(A) + g(B)$.

Naturally, the consequences of Theorem 6 given in [3] and the corollaries of Theorem 7 given in §3 also hold.

Finally, we would like to point out that the results of this note, as well as those of [3], generalize to some algebras other than lattices. All

Figure 1

that is needed is that the algebras be partially ordered, that polynomials and homomorphisms be isotone, and finally that 0 and 1 can be adjoined in a manner permitting the introduction of covers. For instance, lattice ordered semigroups have all these properties, and therefore all the results of this note and of [3] apply to lattice ordered semigroups.

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