ON GENERATING FUNCTIONS OF CLASSICAL POLYNOMIALS

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ABSTRACT. An expansion of an arbitrary power series in terms of products of a power series and a polynomial $p_n^{(\alpha, \beta)}(x)$ (where α, β are not necessarily independent of n) is given.

1. Introduction. Generating functions of the polynomials like $p_n^{\alpha, \beta}(x)$ where α, β are not independent of n has drawn the attention of some investigators like Toscano [10], Feldheim [5], Al-Salam [1], etc. But during the past two or three years there has been a great rush in generalizing and discovering new generating functions for such polynomial sets. On this score Brown [3], Carlitz [4], Srivastava [8], and Zeitlin [11] have proved generalizations of a single result due to Brown [2] in different directions. In this note it is intended to show that most of the results of Brown, Carlitz, Srivastava and Zeitlin follow as special cases of a single known result for generalized hypergeometric series. The note is concluded by proving a generalization of the aforesaid result involving generalized hypergeometric series.

2. Verma [9] had proved

$$\begin{aligned}
 {}_{p+s}F_q \left[\begin{matrix} (a_p), (b_s); xw \\ (c_q) \end{matrix} \right] &= h \sum_{n=0}^{\infty} \frac{[h - n\alpha + 1]_{n-1} [(b_s)]_n [(e_u)]_n}{[1]_n [(c_q)]_n} (-x)^n \\
 (2.1) \quad &\cdot {}_{p+2}F_{u+2} \left[\begin{matrix} (a_p), 1 + h(1 - \alpha)^{-1}, -n; w \\ (e_u), h(1 - \alpha)^{-1}, h - n\alpha + 1 \end{matrix} \right] \\
 &\cdot {}_{s+u+1}F_q \left[\begin{matrix} (b_s) + n, (e_u) + n, h + n(1 - \alpha); x \\ (c_q) + n \end{matrix} \right],
 \end{aligned}$$

provided $p + s \leq q$ or $p + s = 1 + q$ and $|xw| < 1$; $s + u + 1 < q$ or $s + u = q$ and $|x| < 1$, and the series of the hypergeometric functions on the

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right-hand side is absolutely convergent. (This transformation is a slight generalization of an earlier result due to Niblett [7].)

In the above transformation, as usual, $(a_{r,p})$ means the sequence of $(r-p+1)$ parameters $a_r, a_{r+1}, \dots, a_p$ (when $r=1$ write (a_p) instead of $(a_{1,p})$); $[a]_n = a(a+1)(a+2) \cdots (a+n-1)$, $[a]_0 = 1$; $[a]_{-n} = (-)^n/[1-a]_n$ and

$${}_rF_s \left[\begin{matrix} (a_r); z \\ (b_s) \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[(a_r)]_n}{[(b_s)]_n} \frac{z^n}{n!}; \quad r \leq s \text{ or } r = s+1 \text{ and } |z| < 1.$$

In the first instance if in (2.1) we replace p, u, q and s by $p+1, u+1, s+u+2$ and $s+1$, respectively, and set $a_{p+1} = a/(1+b)$, $b_{s+1} = c_{s+1} = a/(1+b)$, $e_{u+1} = 1 + a/(1+b)$, $(c_s) = (b_s)$, $(c_{s+2}, s+u+1) = (e_u)$, $c_{s+u+2} = 1 + a/(b+1)$, $x = t$ and $w = x$, we get the following results due to Srivastava [8, Equation (9)]:

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{a+(b+1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} (\alpha_p), -n; x \\ (\beta_q), 1+a+bn \end{matrix} \right] t^n \\ (2.2) \quad = \frac{(1+v)^{a+1}}{1-bv} {}_pF_q \left[\begin{matrix} (\alpha_p); -xv \\ (\beta_q) \end{matrix} \right], \end{aligned}$$

where $v = t(1+t)^{b+1}$; $v(0) = 0$.

Similarly specializing the parameters in (2.1) the other result of Srivastava [8, Equation (11)] can also be deduced as a special case. Hence the results of Feldheim [5], Carlitz [4] and Brown [2], [3] can be obtained as has been shown by Srivastava [8]. As a matter of fact some of the results of Al-Salam [1] and Toscano [10] can also be obtained as special cases of Srivastava's result (2.2). In particular, if in (2.2) we set $b=1$, replace a by $a-1$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[a+n]_n}{n!} {}_{p+1}F_{q+1} \left[\begin{matrix} (\alpha_p), -n; x \\ (\beta_q), a+n \end{matrix} \right] t^n \\ = \frac{1}{\sqrt{1-4t}} \left(\frac{2}{1+\sqrt{1-4t}} \right)^{a-1} {}_pF_q \left[\begin{matrix} (\alpha_p); \frac{-4xt}{(1+\sqrt{1-4t})^2} \\ (\beta_q) \end{matrix} \right], \end{aligned}$$

which is a slight generalization of a result due to Toscano [10].

However if in (2.2) we set $p=3, q=2, b=0, \alpha_1 = -r, \alpha_2 = \gamma + r + 1$,

$\alpha_3 = 1 + a$, $\beta_1 = 1 + \alpha$, $\beta_2 = 1 + \beta$ and write $w/(1-w)$ for t we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{a+n}{n} {}_3F_2 \left[\begin{matrix} -n, -r, \gamma+r+1; x \\ 1+\alpha, 1+\beta \end{matrix} \right] w^n \\ = (1-w)^{a-1} {}_3F_2 \left[\begin{matrix} -r, \gamma+r+1, a+1; -\frac{xw}{1-w} \\ 1+\alpha, 1+\beta \end{matrix} \right], \end{aligned}$$

which for $a = 0$ reduces to a result due to Al-Salam [1, Equation (6.3)].

Similarly the transformations (6.12) and (6.11) of Al-Salam [1] can also be obtained by specializing the parameters in (2.2). (It may be worth mentioning that (6.12) could be deduced from (6.11) by replacing w by w/λ and letting $\lambda \rightarrow \infty$.)

On the other hand if in (2.1) we set $p = q = 2$, $s = u = 1$, $\alpha = 1 - b$, $h = c$, $w = 1$ and $a_1 = c - a$, $a_2 = c/b$, $e_1 = 1 + (c/b)$, $b_1 = c$, $c_1 = c$, $c_2 = 1 + (c/b)$ and write $-x$ for x , then, on using the following transformation on the left-hand side of the resulting expression

$${}_2F_1 \left[\begin{matrix} a, b; x \\ c \end{matrix} \right] = (1-x)^{-a} {}_2F_1 \left[\begin{matrix} a, c-b; \frac{x}{x-1} \\ c \end{matrix} \right],$$

we get the following elegant transformation of Gould [6]:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{c}{c+bn} \binom{a+bn}{n} \left[\frac{x}{(1+x)^b} \right]^n \\ = (1+x)^a \sum_{n=0}^{\infty} (-1)^n \binom{a-c}{n} \binom{c/b+n}{n}^{-1} \left(\frac{x}{1+x} \right)^n. \end{aligned}$$

However in the above transformation if we replace a and c by $a+br$ and $c+br$, respectively, and sum both the sides from $r = 0$ to ∞ after multiplying both sides by $[(a_p)]_r x^r / [(b_q)]_r [1]_r$, we get, on some simplification, the theorem due to Zeitlin [11, Theorem II].

3. We conclude the note by proving the following generalization of (2.1).

$$\begin{aligned} \sum_{n=0}^{\infty} c_n d_n \frac{x^n w^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \sum_{k=0}^{\infty} \binom{n}{k} c_k [h+k(1-\alpha)] (-w)^k \\ (3.1) \quad \cdot \sum_{s=0}^{\infty} [(e_u)_s + k]_{s+n-k} [h+k+1-n\alpha]_{s+n-k-1} d_{s+n} \frac{x^s}{[1]_s}, \end{aligned}$$

where $\{c_n\}$ and $\{d_n\}$ are arbitrary sequences.

Proof. Equating the coefficients of w^N on both the sides we get

$$\frac{d_N}{N!} = \{h + (1 - \alpha)N\} \sum_{r=0}^{\infty} \frac{(-)^r}{(r+N)!} \sum_{s=0}^{\infty} \frac{[(e_u) + N]_{r+s} [h + N - N\alpha - r\alpha + 1]_{r+s-1}}{[1]_s} d_{N+r+s} x^{s+r}.$$

The coefficient of the term independent of x on the right-hand side is readily seen to be $d_N/N!$. So it only remains to show that for $M > 0$,

$$\{h + (1 - \alpha)N\} \frac{[(e_u) + N]_M}{[1]_M} d_{N+M} \sum_{r=0}^M \frac{[-M]_r [h + N - N\alpha - r\alpha + 1]_{M-1}}{r!} = 0,$$

which follows readily from the fact that $[h + N - N\alpha - r\alpha + 1]_{M-1}$ is a polynomial in r of degree $M-1$ and so its M th difference vanishes. This completes the proof of (3.1).

(3.1) reduces to (2.1) if we set $C_n = [(a_p)]_n / [(e_u)]_n$, $d_n = [(b_s)]_n / [(c_a)]_n$. However it may also be mentioned that the result added in the proof by Srivastava [8], which is attributed to Professor Carlitz, could be deduced as a special case of (3.1).

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