CONTINUOUS SURJECTIONS FROM CANTOR SETS TO COMPACT METRIC SPACES

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ABSTRACT. We present here a new proof of the fact that any compact metric space is the continuous image of the Cantor discontinuum.

It is well known that if C is the Cantor "middle thirds" set and X a nonempty compact metric space, there is a continuous surjection $f: C \rightarrow X$. The two "standard" proofs of this are given in the references below. We present here a third, more direct proof. In view of the fact that any compact, perfect, totally disconnected, metrizable space (called a Cantor set) is homeomorphic to C , it is sufficient to prove the following

Theorem. *Let X be a nonempty compact metric space. Then there is a Cantor set K and a continuous surjection $g: K \rightarrow X$.*

Proof. By an easy inductive argument based on the compactness of X , we can obtain a sequence $\{N_n: n = 1, 2, \dots\}$ of integers greater than 1 and a sequence

$$\mathcal{U}_n = \{U_{i_1, i_2, \dots, i_n}: 1 \leq i_j \leq N_j \text{ for } 1 \leq j \leq n\}$$

of covers of X such that for each n

- (1) each $U \in \mathcal{U}_n$ is the closure of a nonempty open set in X , of diameter less than 2^{-n} ;
- (2) for fixed $i_1, i_2, \dots, i_n$,

$$U_{i_1, i_2, \dots, i_n} = \bigcup \{U_{i_1, i_2, \dots, i_n, i_{n+1}} \in \mathcal{U}_{n+1}\}.$$

Now for each $n = 1, 2, \dots$ let S_n be the discrete topological space consisting of the points $1, 2, \dots, N_n$. Set $K = \prod_{n=1}^{\infty} S_n$, with the product topology. One verifies trivially that K is a Cantor set.

Now define the map $g: K \rightarrow X$ by

$$(3) \quad g(\langle i \rangle) = g(\langle i_1, i_2, \dots \rangle) = U_{i_1} \cap U_{i_1, i_2} \cap U_{i_1, i_2, i_3} \cap \dots.$$

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We have by (1), (2), and the Cantor intersection theorem that g is well defined. g is clearly a surjection, and we now show it to be continuous. For let $x = g(\langle i \rangle)$ as defined in (3), and V be any neighborhood of x . We will produce a neighborhood U of $\langle i \rangle$ with $g(U) \subset V$. Since V is open and the diameters of the $\bigcup_n$ decrease uniformly to zero, we have that there is some N such that $\hat{U} = U_{i_1, i_2, \dots, i_N}$ is contained in V . Consider now the set $U = \{i_1\} \times \{i_2\} \times \dots \times \{i_N\} \times \prod_{n=N+1}^{\infty} S_n$. U is obviously a neighborhood of $\langle i \rangle$; and $g(U) = \hat{U} \subset V$, completing the proof.

REFERENCES

1. John L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955, pp. 165–166. MR 16, 1136.
2. John G. Hocking and Gail S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961, pp. 126–128. MR 23 #A2857.

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