LOCALIZATION PROPERTIES OF BASIC CLASSES OF C^∞ -FUNCTIONS

R. B. DARST

ABSTRACT. Suppose that each of $\mathfrak{M} = C\{M_n\}$ and $\mathfrak{K} = C\{K_n\}$ is a basic class of C^∞ -functions defined on $I = [0, 2\pi]$ and $\mathfrak{M}$ is not a subset of $\mathfrak{K}$. Then it is shown that $\mathfrak{M}$ contains functions which are nowhere locally in $\mathfrak{K}$. One of the corollaries asserts that there are quasi-analytic functions which are nowhere locally analytic. It is also shown that many non-quasi-analytic classes contain functions which are nowhere locally quasi-analytic.

I. Introduction. A basic class, $C\{M_n\}$, of C^∞ -functions defined on $I = [0, 2\pi]$ is determined by a sequence $\{\mu_i\}$ of positive numbers with $1 \geq \mu_1 \geq \mu_2 \geq \dots \rightarrow 0$ as follows. For each positive integer n , denote $(\prod_{j=1}^n \mu_j)^{-1}$ by M_n . Then $f \in C\{M_n\}$ if there is a positive number R_f such that $\|f\|_\infty = \sup\{|f(x)|; x \in I\} \leq R_f$ and $\|f^{(n)}\|_\infty \leq (R_f)^n M_n$ for all $n = 1, 2, \dots$.

The lemma of [2, p. 112] implies that every C^∞ -function is in a basic class. Moreover, it suffices to study basic classes (cf. [2, pp. 112–117]).

A C^∞ -function f defined on I is said to belong to $C\{M_n\}$ locally at x if there is a segment (α, β) containing x and a positive number $P = P(f, [\alpha, \beta])$ such that the restriction, g , of f to $[\alpha, \beta] \cap I$ satisfies $\|g\|_\infty \leq P$ and $\|g^{(n)}\|_\infty \leq P^n M_n$ for all positive integers n .

Both [2, Chapter 5] and [6, Chapter 19] provide the uninitiated reader with a pleasant introduction to these fascinating classes of functions. However, our results are based on deep technical properties established by S. Mandelbrojt in [3].

Recall that a basic class is quasi-analytic if $f \in C\{M_n\}$ and $x \in I$ with $0 = f(x) = f^{(k)}(x)$, $k = 1, 2, \dots$, imply $f \equiv 0$. The Denjoy-Carleman theorem (cf. [6, p. 376]) tells us that $C\{M_n\}$ is a quasi-analytic class if, and only if, $\sum \mu_i = \infty$.

A C^∞ -function is said to be quasi-analytic if it is in a quasi-analytic class (cf. [2, p. 112]).

Received by the editors August 24, 1972.

AMS (MOS) subject classifications (1970). Primary 26A93, 26A90, 30A78.

Key words and phrases. Analytic functions, C^∞ -functions, quasi-analytic functions.

Next, let us consider the following examples:

- (1) $\mathfrak{Q} = C\{n!\}$, i.e., $\mu_n = (n)^{-1}$;
- (2) $\mathfrak{Q} = C\{M_n\}$, where $\mu_n = (n \ln(n+1))^{-1}$;
- (3) $\mathfrak{M}_p = C\{M_n\}$, where $\mu_n = n^{-p}$; $M_n = (n!)^p$, $0 < p < \infty$;
- (4) $\mathfrak{E} = C\{(2n)!\}$, i.e., $\mu_n = (2n-1)(2n)$.

Then [6, p. 374] $\mathfrak{Q} = \mathfrak{M}_1$ is the set of analytic functions on I and (Denjoy-Carleman) $\mathfrak{Q}$ is a quasi-analytic class. Moreover, [3, pp. 63-64] $\mathfrak{M}_1 \subsetneq \mathfrak{Q} \subsetneq \mathfrak{M}_p \subsetneq \mathfrak{M}_q$, $1 < p < q$.* Also [3, p. 76] the function $\exp(-1/x^2)$ belongs to $\mathfrak{E}$, and, since $[(2n-1)2n]/[4n^2] \rightarrow 1$, $\mathfrak{E} = \mathfrak{M}_2$. Thus ([3, pp. 2-7], [4]) $\mathfrak{M}_2$ contains nowhere locally analytic functions. In §IV we shall show that if $p > 1$, then $\mathfrak{M}_p$ contains nowhere locally quasi-analytic functions after we establish the basic result on local comparison of classes in §III. To this latter end, we recall that A. Gorny [1] has shown that $C\{M_n\}$ is a subset of $C\{K_n\}$ if, and only if, $(M_n)^{1/n} = O[(K_n)^{1/n}]$ and we define an appropriate metric on the space, C^∞ , of infinitely differentiable functions defined on the interior $U = (0, 2\pi)$ of I in §II.

II. A metric for C^∞ . First, let $\{U_i\}_{i=1}^\infty$ be a sequence of segments (a_i, b_i) in R which satisfies

- (i) the closure, V_i , of U_i is a compact subset of U ;
- (ii) $\bigcup_i U_i = U$;

and

(iii) if $x \in U$ and W is an open set containing x , then there exists a positive integer i for which $x \in U_i \subset V_i \subset W$.

Denote by (S, ρ) the metric space of sequences of real numbers, where the distance $\rho(x, y)$ between an element $x = \{x_i\}_{i=0}^\infty$ of S and an element $y = \{y_i\}_{i=0}^\infty$ of S is defined to be

$$\rho(x, y) = \sum_{i=0}^{\infty} 2^{-i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

Let T be the metric space of S valued functions defined on U . For $u, v \in T$, let d_i be the semimetric defined by

$$d_i(u, v) = \sup_{x \in V_i} \rho(u(x), v(x)),$$

and then let

$$\delta(u, v) = \sum_{i=1}^{\infty} 2^{-i} d_i(u, v).$$

Then (T, δ) is a complete metric space.

Next, we define a linear map, ϕ , of C^∞ into T by $(\phi f)(x) = (f(x), f^{(1)}(x), \dots)$. Finally, we define $d(f, g) = \delta(\phi(f), \phi(g))$, where f and $g \in C^\infty$. In view of [5, Theorem 7.17] we see that (C^∞, d) is a complete metric space.

III. Local comparison of classes.

Theorem. *Suppose that each of $\mathfrak{M} = C\{M_n\}$ and $\mathfrak{K} = C\{K_n\}$ is a basic class of C^∞ -functions, and $\mathfrak{M}$ is not a subset of $\mathfrak{K}$. Then $\mathfrak{M}$ contains a function which is nowhere locally in $\mathfrak{K}$.*

Proof. Since the set of points at which a C^∞ -function defined on I is locally in $\mathfrak{K}$ is an open subset of I , it suffices to restrict our attention to points in U . Let $\mathfrak{N} = \{f \in \mathfrak{M}; R_f \text{ can be chosen } \leq 1\}$. No confusion should result when we tacitly shift the domains between I and U . Thus, $\mathfrak{N}$ is a closed subset of C^∞ and $(\mathfrak{N}, d)$ is a complete metric space. Denote by $A_{i,n}$ the set of elements f of C^∞ for which the restriction f_i of f to V_i is in $\mathfrak{K}$ with $P(f_i, V_i) \leq n$; $A_{i,n}$ is a closed subset of C^∞ . Since the theorem follows from the Baire category theorem if we can show that $A_{i,n} \cap \mathfrak{N}$ is a nowhere dense subset of $\mathfrak{N}$, let $f \in \mathfrak{N}$ and $\epsilon > 0$; let $\mathfrak{O}$ denote the set of elements g of $\mathfrak{N}$ satisfying $d(f, g) < \epsilon$. We must show that $\mathfrak{O}$ is not a subset of $A_{i,n}$. If $f \notin A_{i,n}$, we are done; otherwise, multiplying the function ϕ defined in [3, p. 56] by $(\Sigma(1/n^2))^{-1}$ gives us a function α : $\alpha(x) = \Sigma_{n \geq 1} c_n \cos nx$, which is in $\mathfrak{N} - \mathfrak{K}$. If α were to belong to $\mathfrak{K}$ locally at each point of the compact set I , then α would belong to $\mathfrak{K}$. Hence, there exists a point x of U such that α does not belong to $\mathfrak{K}$ locally at x . Choose $y \in U_i$ and let $\beta(t) = \alpha(t + (x - y))$, $t \in R$. Then for sufficiently small positive numbers a , the function $(1 - a)f + a\beta$ belongs to $(\mathfrak{N} - A_{i,n}) \cap \mathfrak{O}$; thus, the theorem is established.

Several corollaries are immediate; two such follow.

Corollary 1. *If $C\{M_n\}$ contains a nonanalytic function, then $C\{M_n\}$ contains a nowhere locally analytic function.*

Corollary 2. *If $C\{M_n\}$ is not a quasi-analytic class and $C\{N_j\}$ is a quasi-analytic class, then $C\{M_n\}$ contains a function which is nowhere locally in $C\{N_j\}$.*

The following observation yields other corollaries, one of which we shall transcribe.

For each sequence $1 \geq \mu_1 \geq \mu_2 \geq \dots \rightarrow 0$, there is a sequence $1 \leq$

$a_1 < a_2 < \dots \rightarrow \infty$ such that (i) if $\sum \mu_i < \infty$, then $1 \geq a_1 \mu_1 \geq a_2 \mu_2 \geq \dots \rightarrow 0$ and $\sum a_n \mu_n < \infty$, or (ii) if $\sum \mu_n = \infty$, then $\sum (\mu_n / a_n) = \infty$.

Corollary 3. *If $C\{M_n\}$ is a quasi-analytic class, then there is a quasi-analytic function which is nowhere locally in $C\{M_n\}$.*

Notice that Corollary 2 does not assert that a non-quasi-analytic class $\mathfrak{M}$ contains nowhere locally quasi-analytic functions. In the next section we give an explicit construction to show that for $p > 1$, $\mathfrak{M}_p$ contains nowhere locally quasi-analytic functions.

IV. $\mathfrak{M}_p$, $p > 1$. To motivate our approach, let us begin with some general remarks. First (cf. [6, p. 375]) any non-quasi-analytic class contains functions with small support. Moreover, H. E. Bray [3, pp. 79–84] gave an explicit construction for such functions. However, Bray's functions are not quasi-analytic on the interiors of their supports. Our construction will be a modification of that of L. E. May [4]; his construction uses $\exp(-1/x^2)$, and this function is analytic on $(0, \infty)$. To verify that $\mathfrak{M}_p$ contains appropriate functions, we establish the following

Lemma. *For each positive integer k , there is a positive number R_k such that $f_k = \exp(-x^{-k}) \in \mathfrak{M}_{1+1/k}$ with $R_k = R_{f_k}$.*

Proof. Fix k and denote f_k by f . Then for $x > 0$, we may estimate $f^{(n)}(x)$ by using Cauchy's integral formula as follows. Denote by C_x the circle with center x and radius $x \sin \alpha$, where $\alpha = (2\pi/3k)$; C_x is the largest circle centered at x such that $k\theta \leq 2\pi/3$ whenever $z = re^{i\theta}$ belongs to C_x . On C_x , $|f(z)| = \exp(-\cos(k\theta)/r^k) \leq \exp(-1/(bx^k))$, where $b = 2(1 + \sin \alpha)^k$. Hence, $|f^{(n)}(x)| \leq n!(\sin \alpha)^{-n} x^{-n} \exp(-1/(bx^k))$. Differentiate the right side of the preceding inequality with respect to x and find that its maximum occurs when $nbx^k = k$; then substitute and apply Stirling's formula to obtain R_k such that $|f^{(n)}(x)| \leq (R_k)^n (n!)^{(1+1/k)}$ for all $x > 0$.

Theorem. *Suppose $p > 1$. Then $\mathfrak{M}_p$ contains a nowhere locally quasi-analytic function.*

Proof. Choose k with $1 + 1/k < p$. Referring to the Lemma, notice that if $g(x) = f(cx)$, then $g^{(n)}(x) = c^n f^{(n)}(x)$; thus, we choose $c = 1/(2R_k)$, so that $R_g = 1/2$. Then we recall that we can let $g(x) = 0$, $x < 0$, and define $h(t) = h(a, b, t) = g(t-a)g(b-t)$, $t \in R$, to obtain [6, pp. 373–374] h in $\mathfrak{M}_p$ with $R_h = 1$. Moreover, the support of h is $[a, b]$, h is positive on (a, b) , and h is analytic on (a, b) . Let r_i be a countable dense subset of $(0, 2\pi)$

and set

$$\begin{aligned}\phi(x) &= \sum_{j=1}^{\infty} (2^j M_j)^{-1} \sum_{i=1}^{\infty} (2^i M_i)^{-1} h(r_i - 2^{-(i+j)}, r_i + 2^{-(i+j)}, x) \\ &= \sum_{i,j \geq 1} (2^{i+j} M_i M_j)^{-1} h(r_i - 2^{-(i+j)}, r_i + 2^{-(i+j)}, x).\end{aligned}$$

A close look at the argument given in [4] establishes the theorem.

REFERENCES

1. A. Gorny, *Contribution à l'étude des fonctions dérivables d'une variable réelle*, Acta Math. **71** (1939), 317–358. MR **1**, 137; 400.
2. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR **40** #1734.
3. S. Mandelbrojt, *Analytic functions and infinitely differentiable functions*, Rice Institute Pamphlet **29**, no. 1, 1942, Houston, Texas. MR **3**, 292.
4. L. E. May, *On C^∞ functions analytic on sets of small measure*, Canad. Math. Bull. **12** (1969), 25–30. MR **39** #2933.
5. W. Rudin, *Principles of mathematical analysis*, 2nd ed., McGraw-Hill, New York, 1964. MR **29** #3587.
6. ———, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR **35** #1420.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS,
COLORADO 80521