NECESSARY AND SUFFICIENT CONDITIONS FOR FACTORABILITY OF NONNEGATIVE OPERATOR- VALUED FUNCTIONS ON A BANACH SPACE

A. G. MIAMEE AND H. SALEHI¹

ABSTRACT. Several necessary and sufficient conditions for factorability of positive operator-valued functions on a Banach space are given. These characterizations involve the analysis of quasi square root and their corresponding invariant subspaces. The work extends to the Banach space a recent theorem of Yu. A. Rozanov and a certain earlier result of R. Payen. It also reveals the connection between their works on the factorization problem.

1. Introduction. The problem of factoring nonnegative operator-valued functions on a Banach space plays an important role in the study of Banach space valued stationary stochastic processes [1]. In [4] we were able to extend to the Banach space case the work of R. G. Douglas [2] on factoring nonnegative operator-valued functions. However these results provide only sufficient conditions for the factorization problem. Our purpose here is to establish some necessary and sufficient conditions for factorability of nonnegative operator-valued functions on a Banach space. This extends to the Banach space the recent work of Yu. A. Rozanov [6] and a certain earlier result of R. Payen [5] on necessary and sufficient conditions for the factorization problem. It also reveals the close connection which exists between these characterizations.

When f is a positive operator-valued function on a Hilbert space, $\sqrt{f}$, the square root of f whose existence is known, plays a significant role. But when f is a positive operator-valued function on a Banach space $\mathcal{X}$, the existence of a square root in the ordinary way does not make sense. Nevertheless, we have recently proved [4] the existence of a measurable operator-valued function Q on $\mathcal{X}$ into some auxiliary Hilbert space which behaves

Received by the editors August 14, 1973.

AMS (MOS) subject classifications (1970). Primary 42A88, 42A96; Secondary 60G10.

Key words and phrases. Banach space, positive operator-valued functions, quasi square root, factorability, regularity, stationary processes.

¹Supported in part by NSF GP-28658.

Copyright © 1974, American Mathematical Society

almost like a square root in the sense that $f = Q^*Q$. The operator-valued function Q is called a quasi square root.

In §2 we set up necessary terminology and state some known results. In §3 we prove our main theorem on characterizing factorable positive operator-valued functions on a Banach space. In establishing our result we make use of quasi square root and techniques employed in [6].

2. Preliminaries. Let K be a separable Hilbert space and let $L_2(K)$ denote the Hilbert space of all K -valued functions defined on the unit circle that are weakly measurable and have square summable norms with respect to (w.r.t.) Lebesgue measure. The $L_2(K)$ inner product of two functions g_1 and g_2 is given by

$$\frac{1}{2\pi} \int_0^{2\pi} (g_1(e^{i\theta}), g_2(e^{i\theta})) d\theta.$$

The operator U , defined to be multiplication by $e^{i\theta}$ on $L_2(K)$, is unitary and a shift. The subspace $L_2^{0+}(K)$ ($L_2^{0-}(K)$) consisting of those functions g in $L_2(K)$ for which $\int_0^{2\pi} g(e^{i\theta}) e^{in\theta} d\theta = 0$ for $n > 0$ ($n < 0$) is an invariant subspace of U (U^*).

Let $\mathcal{X}$ be a Banach space. $\mathcal{X}^*$ denotes the space of all bounded conjugate linear functionals on $\mathcal{X}$. For any two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $B(\mathcal{X}, \mathcal{Y})$ denotes the class of all bounded linear operators on $\mathcal{X}$ into $\mathcal{Y}$. A weakly measurable $B(\mathcal{X}, K)$ -valued function $A = A(e^{i\theta})$ is said to be analytic if for each $x \in \mathcal{X}$, $A(e^{i\theta})x \in L_2^{0-}(K)$. From now on it is assumed that the Banach space $\mathcal{X}$ is separable.

2.1. Definition. Let f be in $B(\mathcal{X}, \mathcal{X}^*)$. f is said to be nonnegative if for each x in $\mathcal{X}$, $(f(x))(x) \geq 0$. $B^+(\mathcal{X}, \mathcal{X}^*)$ denotes the class of all nonnegative operators on $\mathcal{X}$ into $\mathcal{X}^*$.

2.2. Definition. Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. We say that f is factorable if there exists a Hilbert space K and an analytic $B(\mathcal{X}, K)$ -valued function $A = A(e^{i\theta})$ such that $f(e^{i\theta}) = A^*(e^{i\theta})A(e^{i\theta})$, in the sense that $(f(e^{i\theta})x)(y) = (A(e^{i\theta})x, A(e^{i\theta})y)$; $x, y \in \mathcal{X}$.

We now state the following crucial lemma which is proved in [4].

2.3. Lemma. Let f be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then there exists a separable Hilbert space K and a measurable $B(\mathcal{X}, K)$ -valued function $Q = Q(e^{i\theta})$ on the unit circle such that

$$f(e^{i\theta}) = Q^*(e^{i\theta})Q(e^{i\theta}),$$

in the sense that

$$(f(e^{i\theta})x)(y) = (Q(e^{i\theta})x, Q(e^{i\theta})y); \quad x, y \in \mathcal{X}.$$

The above lemma allows us to make the following

2.4. Definition. The function $Q = Q(e^{i\theta})$ in Lemma 2.3 is called a quasi square root of f .

In the absence of the square root, a quasi square root has almost all the desired properties. One of its applications is demonstrated in Theorem 3.8 where necessary and sufficient conditions for factorability of f are given.

We now make a digression to state some necessary terminologies concerning Banach space valued stationary stochastic processes. Let $\mathcal{X}$ be a separable Banach space and $\mathcal{K}$ be a separable Hilbert space. By a stationary stochastic process (SSP) we mean a $B(\mathcal{X}, \mathcal{K})$ -valued sequence ξ_n , $-\infty < n < \infty$, of bounded linear operators on $\mathcal{X}$ into $\mathcal{K}$ such that $\xi_n^* \xi_m$ depends only on $m - n$. The operator $R(m - n) = \xi_n^* \xi_m$ is called the covariance operator of the process. It is known [1] that we have the representation

$$R(m - n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(m-n)\theta} dF(e^{i\theta}),$$

where the spectral distribution F is a measure on the Borel sets of the unit circle with values in $B^+(\mathcal{X}, \mathcal{X}^*)$, and the above integral is taken in the weak sense. When F is absolutely continuous (a.c.) w.r.t. Lebesgue measure, its density f is a bilinear functional on $\mathcal{X} \times \mathcal{X}$. f is called the spectral density of the process. We denote by $H_\xi(n)$ the subspace of $\mathcal{K}$ spanned by all $\xi_k x$, $-\infty < k \leq n$, $x \in \mathcal{X}$ and set $H_\xi(-\infty) = \bigcap_n H_\xi(n)$. We say that the process is regular if $H_\xi(-\infty) = \{0\}$.

Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathcal{X}, \mathcal{X}^*)$ -valued function on the unit circle. Then by Lemma 2.3 a quasi square root of f , $Q = Q(e^{i\theta})$, with values in $B(\mathcal{X}, \mathcal{K})$ exists. Let $\xi_n = e^{in\theta} Q(\theta)$. Then ξ_n , $-\infty < n < \infty$, is a $B(\mathcal{X}, L_2(\mathcal{K}))$ -valued SSP whose spectral density is f . From here on ξ_n , $-\infty < n < \infty$, represents this process.

In §3 we need Lemma 2.5 contained in [6], which can also be essentially obtained from Theorem 8 of [3]. First we introduce some notations (c.f. [6]). Let B be a linear manifold in $L_2(\mathcal{K})$ and $S = \{g_n(e^{i\theta})\}_{n=1}^\infty$ be a complete orthonormal system of functions in B . We denote by $B_\delta(e^{i\theta})$ the linear manifold in the Hilbert space $\mathcal{K}$ generated by all values $g_1(e^{i\theta})$, $g_2(e^{i\theta})$, $\dots$. Obviously the closure $\bar{B}(e^{i\theta}) = \bar{B}_\delta(e^{i\theta})$ does not depend on δ in the sense that $\bar{B}_{\delta_1}(e^{i\theta}) = \bar{B}_{\delta_2}(e^{i\theta})$ a.e. θ , if δ_1 and δ_2 are any two complete orthonormal systems in B .

In case $B = Q(e^{i\theta})\mathfrak{X}$ it easily follows that a.e.

$$\overline{B}(e^{i\theta}) = \overline{Q(e^{i\theta})\mathfrak{X}} \quad \text{a.e.,}$$

where $\overline{Q(e^{i\theta})\mathfrak{X}}$ denotes the closure of the range of the operator $Q(e^{i\theta})$.

2.5. Lemma. *Let B be a linear manifold in $L_2(\mathcal{K})$. Then the subspace $\mathfrak{S}(e^{in\theta}B, -\infty < n < \infty)$ generated by $e^{in\theta}B, -\infty < n < \infty$, consists of all functions $g \in L_2(\mathcal{K})$ such that $g(e^{i\theta}) \in \overline{B}(e^{i\theta})$ a.e.*

3. Main results. In this section we prove our main results. $\mathfrak{X}$ is a separable Banach space and $\mathcal{K}$ is a separable Hilbert space. f is a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ -valued function on the unit circle. Q will denote a quasi square root of f with values in $B(\mathfrak{X}, \mathcal{K})$. $\xi_n = e^{in\theta}Q(e^{i\theta}), -\infty < n < \infty$, is a $B(\mathfrak{X}, L_2(\mathcal{K}))$ -valued SSP with the spectral density f . Let $H = \mathfrak{S}(\xi_n x, x \in \mathfrak{X}, -\infty < n < \infty)$ and $H(n) = \mathfrak{S}(\xi_n x, x \in \mathfrak{X}, -\infty \leq n)$.

We shall be interested in the structure of the subspaces $B = H(T) \ominus H(S)$, where T, S are some sets of integers and for any set T , $H(T) = \mathfrak{S}(\xi_n x, x \in \mathfrak{X}, n \in T)$. One can say that B is the innovation of $H(T)$ in comparison with $H(S)$.

L_T will denote the linear space of all $\mathfrak{X}^*$ -valued integrable functions ϕ {for each $x \in \mathfrak{X}$, $\phi(e^{i\theta})x$ is summable} with Fourier decomposition of the form

$$(3.1) \quad \phi(e^{i\theta}) \sim \sum_{n \in T} a_n e^{ni\theta}, \quad a_n \in \mathfrak{X}^*,$$

i.e.

$$\phi(e^{i\theta})x \sim \sum_{n \in T} a_n x e^{ni\theta}, \quad x \in \mathfrak{X},$$

such that

$$(3.2) \quad \phi(e^{i\theta}) \in Q^*(e^{i\theta})\mathcal{K},$$

and

$$(3.3) \quad \int_0^{2\pi} \|Q^{*-1}(e^{i\theta})\phi(e^{i\theta})\|^2 d\theta < \infty,$$

where $Q^{*-1}(e^{i\theta})$ is the inverse operator from $Q^*(e^{i\theta})\mathcal{K}$ onto $\mathcal{N}_{Q^*(e^{i\theta})}^\perp =$ orthogonal complement of the null subspace of $Q^*(e^{i\theta})$.

3.4. Lemma. *Let S be the complement of T in the set of all integers and $B_T = H(T) \ominus H(S)$. Then*

$$(3.5) \quad B_T = Q^{*-1}L_T.$$

Proof. Let $\phi(e^{i\theta}) \in B_T$. Define $\Psi(e^{i\theta}) = Q^*(e^{i\theta})\phi(e^{i\theta})$. From the relations

$$\begin{aligned} \int_0^{2\pi} (Q^*(e^{i\theta})\phi(e^{i\theta}))(x) d\theta &= \int_0^{2\pi} (\phi(e^{i\theta}), Q(e^{i\theta})x) d\theta \\ &\leq \left(\int_0^{2\pi} \|\phi(e^{i\theta})\|^2 d\theta \right)^{1/2} \left(\int_0^{2\pi} \|Q(e^{i\theta})x\|^2 d\theta \right)^{1/2} < \infty, \end{aligned}$$

it follows that $\Psi(e^{i\theta})(x)$, $x \in \mathfrak{X}$, is summable. Also we have that

$$\begin{aligned} 0 &= \int_0^{2\pi} e^{-is\theta} (\phi(e^{i\theta}), Q(e^{i\theta})x) d\theta \\ &= \int_0^{2\pi} e^{-is\theta} (Q^*(e^{i\theta}), \phi(e^{i\theta}))(x) d\theta, \quad s \in \mathcal{S}. \end{aligned}$$

Since $\phi \in H$, by Lemma 2.5 and the paragraph preceding it, $\phi(e^{i\theta}) \in Q(e^{i\theta})\mathfrak{X}$ a.e. Hence

$$\phi(e^{i\theta}) = Q^{*-1}(e^{i\theta})\Psi(e^{i\theta}) \quad \text{and} \quad \Psi(e^{i\theta}) \in L_T.$$

Thus $B_T \subseteq Q^{*-1}L_T$. Now let $\Psi(e^{i\theta})$ be in L_T . Set $\phi(e^{i\theta}) = Q^{*-1}(e^{i\theta})\Psi(e^{i\theta})$. We will show that $\phi(e^{i\theta}) \in B_T$. We note that for each $x \in \mathfrak{X}$, $(\phi(e^{i\theta}), Q(e^{i\theta})x) = \Psi(e^{i\theta})(x)$. Hence

$$\int_0^{2\pi} e^{-is\theta} (\phi(e^{i\theta}), Q(e^{i\theta})x) d\theta = 0, \quad s \in \mathcal{S}, \quad x \in \mathfrak{X}.$$

It suffices to show that $\phi \in H = \overline{\mathfrak{G}(e^{in\theta}Q(e^{i\theta})\mathfrak{X})}$, $-\infty < n < \infty$. But $\phi(e^{i\theta}) \in$ range of $Q^{*-1}(e^{i\theta}) = Q(e^{i\theta})\mathfrak{X}$. Hence by Lemma 2.5 and the paragraph preceding it, we have that $\phi(e^{i\theta}) \in B_T$. This completes the proof of $Q^{*-1}L_T \subseteq B_T$.

3.6. Corollary. *The relation $\mathfrak{G}(e^{in\theta}B_T, -\infty < n < \infty) = H$ holds if and only if*

$$(3.7) \quad \overline{Q(e^{i\theta})\mathfrak{X}} = \overline{Q^{*-1}L_T(e^{i\theta})} \quad \text{a.e.}$$

Proof. Since $H = \mathfrak{G}(e^{in\theta}Q(e^{i\theta})\mathfrak{X}, -\infty < n < \infty)$, by Lemma 2.5, $H = \mathfrak{G}(e^{in\theta}B_T, -\infty < n < \infty)$ if and only if $\overline{\overline{B}_T}(e^{i\theta}) = \overline{Q(e^{i\theta})\mathfrak{X}}$. But by the last lemma $\overline{\overline{B}_T}(e^{i\theta}) = \overline{Q^{*-1}L_T(e^{i\theta})}$.

We note that this result in particular is useful for regularity if we take T to be the set of all nonnegative integers. In this case the space L_T is included in the Hardy class of functions $\phi(e^{i\theta}) \sim \sum_0^\infty a_n e^{in\theta}$. More precisely they are so that for each $x \in \mathfrak{X}$, the scalar-valued function $\phi(e^{i\theta})(x)$ is in the classical Hardy class H^1 . We also note that in this case the rela-

tion $\mathfrak{G}(e^{in\theta}B_T, -\infty < n < \infty) = H$ of Corollary 3.6 is equivalent to the regularity of the process $\xi_n = e^{in\theta}Q(e^{i\theta}), -\infty < n < \infty$.

We are now ready to prove our main result.

3.8. Theorem. *Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ -valued function on the unit circle. Then the following statements are equivalent:*

1. f is factorable.

2. There exists an analytic $B(\mathfrak{X}, \mathbb{K})$ -valued function Ψ such that

$$(i) \Psi^*(e^{i\theta})\mathbb{K} \subseteq Q^*(e^{i\theta})\mathbb{K};$$

$$(ii) \overline{Q^{*-1}(e^{i\theta})\Psi^*(e^{i\theta})\mathbb{K}} = \overline{Q(e^{i\theta})\mathfrak{X}} \text{ a.e.};$$

$$(iii) \int_0^{2\pi} \|Q^{*-1}(e^{i\theta})\Psi^*(e^{i\theta})k\|^2 d\theta < \infty, k \in \mathbb{K}.$$

3. The process $e^{in\theta}Q(e^{i\theta}), -\infty < n < \infty$, is regular.

4. There exists a sequence $\{\phi_n(e^{i\theta})\}_{n=1}^\infty$ of measurable $\mathbb{K}$ -valued functions such that

$$(a) \text{ a.e. } \theta, \{\phi_n(e^{i\theta})\}_{n=1}^\infty \text{ is an orthonormal basis for } \overline{Q(e^{i\theta})\mathfrak{X}};$$

(b) for each n , $g_n(e^{i\theta})x = Q^*(e^{i\theta})\phi_n(e^{i\theta})x$ is in the usual Hardy class H^2 for all $x \in \mathfrak{X}$.

Proof. (1) $\Rightarrow$ (2). Let $f = \Phi^*\Phi$, where Φ is analytic. Then

$$(3.9) \quad \|Q(e^{i\theta})x\|^2 = \|\Phi(e^{i\theta})x\|^2, \quad x \in \mathfrak{X}.$$

Define

$$(3.10) \quad V(e^{i\theta})Q(e^{i\theta})x = \Phi(e^{i\theta})x, \quad x \in \mathfrak{X}.$$

By (3.9)–(3.10), $V(e^{i\theta})$ can be extended to an isometry on $\overline{Q(e^{i\theta})\mathfrak{X}}$ onto $\overline{\Phi(e^{i\theta})\mathfrak{X}}$. Then the operator-valued function $\Psi = VQ$ satisfies the conditions (i)–(iii) of (2).

(2) $\Rightarrow$ (3). Let Ψ be an analytic $B(\mathfrak{X}, \mathbb{K})$ -valued function satisfying (i)–(iii). By Corollary 3.6 and the paragraph following this corollary, it suffices to show that $\overline{Q(e^{i\theta})\mathfrak{X}} = \overline{Q^{*-1}L_T(e^{i\theta})}$ a.e. Clearly the right-hand side is a subset of the left-hand side. The other inclusion follows from (ii)–(iii) and the fact that for each $k \in \mathbb{K}$, $\Psi^*k \in L_T$.

(3) $\Rightarrow$ (4). Since $e^{in\theta}Q(e^{i\theta}), -\infty < n < \infty$, is a regular process, $H(0) = \mathfrak{G}(e^{in\theta}Q(e^{i\theta})\mathfrak{X}, n \leq 0)$ does not contain a nontrivial doubly invariant subspace of $\mathbb{K}$. Hence by [3, p. 61] it is of the form $VL_2^{0-}(\mathcal{H})$, where V is a measurable isometry operator on some Hilbert space $\mathcal{H}$ into $\mathbb{K}$. Let $\{\phi_n(e^{i\theta})\}_{n=1}^\infty$ be an orthonormal basis for $H(0) \ominus H(-1)$. An argument similar to one used in [5, p. 380] and [3, p. 61] may be used to show that for

almost all θ , $\{\phi_n(e^{i\theta})\}_{n=1}^\infty$ forms an orthonormal basis for $\overline{Q(e^{i\theta})\mathfrak{X}}$. Let $x \in \mathfrak{X}$. To show $g_n(e^{i\theta})(x) = Q^*(e^{i\theta})\phi_n(e^{i\theta})(x)$ is in the Hardy class H^2 , we observe that for each n , $\phi_n \perp e^{ik\theta}Q(e^{i\theta})x$ for $x \in \mathfrak{X}$ and $k \leq -1$. Hence

$$\begin{aligned} \int_0^{2\pi} e^{-ik\theta} g_n(e^{i\theta})(x) d\theta &= \int_0^{2\pi} e^{-ik\theta} Q^*(e^{i\theta})\phi_n(e^{i\theta})(x) d\theta \\ &= \int_0^{2\pi} e^{-ik\theta} (\phi_n(e^{i\theta}), Q(e^{i\theta})x) d\theta = 0. \end{aligned}$$

Thus $g_n x \in H^2$.

(4) $\Rightarrow$ (1). Let (a)–(b) hold. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{K}$. We define the operator-valued function Φ by

$$(3.11) \quad \Phi(e^{i\theta})x = \sum_n \overline{g_n(e^{i\theta})(x)} e_n, \quad x \in \mathfrak{X}.$$

We note that for all $x, y \in \mathfrak{X}$,

$$\begin{aligned} (fx)(y) &= (Qx, Qy) = \sum_n (Qx, \phi_n) \overline{(Qy, \phi_n)} \\ (3.12) \quad &= \sum_n \overline{(Q^* \phi_n x)} (Q^* \phi_n y) = \sum_n \overline{(g_n x)} (g_n y) \\ &= \left(\sum_n \overline{(g_n x)} e_n, \sum_n (g_n y) e_n \right) = (\Phi x, \Phi y), \end{aligned}$$

where the second equality follows by (a). Hence $\Phi(e^{i\theta}) \in B(\mathfrak{X}, \mathcal{K})$.

Because of weak summability of f it follows that for each $x \in \mathfrak{X}$, $\Phi x \in L_2(\mathcal{K})$.

Using (3.11) and (3.12) it is not hard to show that the analytic $B(\mathfrak{X}, \mathcal{K})$ -

valued sequence $\Phi_N(e^{i\theta})x = \sum_{n=1}^N g_n(e^{i\theta})(x) e_n$ converges to $\Phi(e^{i\theta})x$ in $L_2(\mathcal{K})$. Therefore $\Phi \in L_2^0(\mathcal{K})$. Since by (3.12), $f = \Phi^* \Phi$, the result follows.

We remark that if one assumes $\int_0^{2\pi} \|f(e^{i\theta})\| d\theta < \infty$, then part (b) of (4) in the above theorem can be replaced by (b'): For each n , $g_n(e^{i\theta}) = Q^*(e^{i\theta})\phi_n(e^{i\theta})$ satisfies $\int_0^{2\pi} \|g_n(e^{i\theta})\|^2 d\theta < \infty$.

REFERENCES

1. S. A. Cobanjan, *Regularity of stationary processes with values in a Banach space and factorization of operator-valued functions*, Sakhath. SSR Mech. Akad. Moambe 61 (1971), 29–32. (Russian) MR 44 #7631.
2. R. G. Douglas, *On factoring positive operator functions*, J. Math. Mech. 16 (1966), 119–126. MR 35 #782.

3. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964. MR 30 #1409.

4. A. G. Miamee and H. Salehi, *Factorization of positive operator valued functions on a Banach space*, Indiana Univ. Math. J. (to appear).

5. R. Payen, *Fonctions aléatoires du second ordre à valeurs dans un espace de Hilbert*, Ann. Inst. H. Poincaré Sect. B 3 (1967), 323–396. MR 37 #6993.

6. Ju. A. Rozanov, *Some approximation problems in the theory of stationary processes*, J. Multivariate Anal. 2 (1972), 135–144. MR 46 #4597.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823 (Current address of H. Salehi)

Current address (A. G. Miamee): Department of Mathematics, Arya-Mehr University of Technology, Tehran, Iran