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ON THE HOMOLOGY OF FINITE CYCLIC COVERINGS OF HIGHER-DIMENSIONAL LINKS

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ABSTRACT. We produce an explicit formula for the betti numbers of the k-fold branched cyclic covering of a link, in terms of complex kth roots of unity which are also roots of the polynomial invariants of the link. More information is obtained when k is a prime power.

I. Introduction. For classical knots and links, there is a well-known formula [2], [3], [8] for computing the 1-dimensional betti numbers of the k-fold branched cyclic covering space. The formula is given in terms of complex kth roots of unity which are also roots of the polynomial invariants of the infinite cyclic covering space of the complement. This paper gives a new proof of these results, and extends them to a calculation of all the betti numbers for a higher-dimensional link. The proof exploits a long exact sequence [5] relating the homology of the infinite cyclic cover to that of the k-fold unbranched cyclic cover.

The formulas are particularly simple and interesting in the case of prime power coverings (see Theorem 3 and Corollary 5). Moreover, the method of proof of Theorem 3 has turned out to be useful in the study of the monodromy of plane algebraic curves [7].

II. Rational homology invariants. An n-link of multiplicity μ denotes a smooth oriented submanifold L^n_μ of S^{n+2} homeomorphic to μ disjoint copies of S^n. If X denotes the complement of the link, then Alexander duality gives us

$$\begin{aligned} H_i(X;\ Z) &= \mu Z, && i = 1, \\ &= (\mu - 1)Z, && i = n + 1, \\ &= Z, && i = 0, \\ &= 0, && \text{otherwise,} \end{aligned}$$

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where αZ denotes the direct sum of α copies of Z.

The orientation of the manifold pair (S^{n+2}, L^n_μ) allows us to define the homological linking number $\langle\tau, L^n_\mu\rangle$ between 1-cycles τ and the link L^n_μ. This determines a homomorphism

$$\pi_1(X) \xrightarrow{\phi} J(t), \qquad \tau \mapsto t^{\langle\tau, L^n_\mu\rangle}$$

where $J(t)$ is the infinite cyclic multiplicative group on the generator t. The infinite cyclic covering space $\tilde{X}$ of X is the regular covering space associated with Ker ϕ. $\tilde{X}$ is an invariant of oriented link type.

Let Z, Q and C denote the integers, rationals and complex numbers. Let Λ, Γ and Ψ denote the integral, rational and complex group ring of $J(t)$, respectively. We have $\Gamma \cong \Lambda \otimes_Z Q$ and $\Psi \cong \Gamma \otimes_Q C$ and regard $\Lambda \subset \Gamma \subset \Psi$ as subrings. Λ is a Noetherian unique factorization domain, and Γ and Ψ are principal ideal domains. Since $J(t)$ acts on $\tilde{X}$ as the group of covering translations, we have that $H_q(\tilde{X}; Z)$ is a finitely-presented Λ-module for all q, and likewise for $H_q(\tilde{X}; Q)$ and $H_q(\tilde{X}; C)$ over Γ and Ψ. The polynomial invariants are the invariants of the Γ-structure of $H_q(\tilde{X}; Q)$. Since Γ is a PID, we have that $H_q(\tilde{X}; Q) \cong_\Gamma F_q \oplus T_q$ where F_q is Γ-free and T_q is Γ-torsion. If $\lambda \in \Gamma$, let Γ/λ denote the cokernel of multiplication by λ on Γ. It is shown in [5] that in the range $n \geq 2$ or $n = 1$, $\mu = 1$ that

$$\begin{aligned} F_q &= (\mu - 1)\Gamma, && q = 1, n+1, \\ &= 0, && \text{otherwise,} \end{aligned}$$

and $T_q \cong \Gamma/\lambda^q_1 \oplus \cdots \oplus \Gamma/\lambda^q_{m_q}$ where $\lambda^q_i \in \Lambda$, $\lambda^q_{i+1} | \lambda^q_i$ in Λ, $1 \leq i \leq (m_q - 1)$. Moreover, if $q \geq 1$ then $\lambda^q_i(1) = \pm 1 \forall i$; and $\lambda^0_1(t) = (t-1)$. The rank of F_q together with the invariant factors $\{\lambda^q_i\}$ are a complete set of invariants for the Γ-module $H_q(\tilde{X}; Q)$, hence are invariants of oriented link type.

Moreover, the Γ-structure of $H_q(\tilde{X}; Q)$ determines the Ψ-structure of $H_q(\tilde{X}; C) \cong H_q(\tilde{X}; Q) \otimes_Q C$. That is, for $1 \leq q \leq n$, the Ψ-torsion summand of $H_q(\tilde{X}; C)$ is $\Psi/\lambda^q_i \oplus \cdots \oplus \Psi/\lambda^q_{m_q}$, and the Ψ-rank of

$$\begin{aligned} H_q(\tilde{X}; C) &= (\mu - 1), && q = 1, n+1, \\ &= 0, && \text{otherwise.} \end{aligned}$$

The k-fold unbranched cyclic cover X^u_k of X is the regular covering associated with the canonical epimorphism

$$\begin{tikzcd} \pi_1(X) \arrow[rr, "\varphi_k"] \arrow[dr, "\phi"'] & & J_k \\ & J(t) \arrow[ur] & \end{tikzcd}$$

where J_k denotes the finite cyclic multiplicative group of order k. The unbranched cover has a unique completion, the associated k-fold branched cyclic cover X_k^b, branched along the link L_μ^n. Let k be fixed, and let α_i^q be the number of distinct kth roots of unity which are roots of λ_i^q. Let $c_q = \sum_{i=1}^{m_q} \alpha_i^q$, and let $\beta_q = q$-dimensional betti number of X_k^u. We have the following

Theorem 1. *$n \geq 2$ or $n = 1$, $\mu = 1$. Then*

$$\begin{aligned} \beta_q &= c_q + c_{q-1}, & 2 \leq q \leq n, \\ &= k(\mu - 1) + c_1 + 1, & q = 1, \\ &= k(\mu - 1) + c_n, & q = n + 1, \\ &= 1, & q = 0. \end{aligned}$$

Proof. We have that $\beta_q = \operatorname{rank}_Q (H_q(X_k^u; Q)) = \operatorname{rank}_C (H_q(X_k^u; C))$, so it will suffice to compute the complex rank. As in [5], we have the short exact sequence of chain complexes (over Ψ and C):

$$0 \to C_*(\widetilde{X}; C) \xrightarrow{(t^k - 1)} C_*(\widetilde{X}; C) \to C_*(X_k^u; C) \to 0.$$

This induces the long exact sequence of homology

$$\text{(1)} \qquad \cdots \to H_q(\widetilde{X}; C) \xrightarrow{(t^k - 1)} H_q(\widetilde{X}; C) \to H_q(X_k^u; C) \xrightarrow{\partial} \cdots.$$

Let Ker_q and Cok_q denote the kernel and cokernel, respectively, of the map $(t^k - 1)$; then the following is an exact sequence of C-vector spaces:

$$0 \to \operatorname{Ker}_q \to H_q(\widetilde{X}; C) \xrightarrow{(t^k - 1)} H_q(\widetilde{X}; C) \to \operatorname{Cok}_q \to 0.$$

Note that in the range $2 \leq q \leq n$, $H_q(\widetilde{X}; C)$ is pure Ψ-torsion, so is a finite-dimensional C-vector space, and hence $\dim_C (\operatorname{Ker}_q) = \dim_C (\operatorname{Cok}_q)$. The long exact sequence (1) decomposes into a number of short exact sequences

$$\text{(2)} \qquad 0 \to \operatorname{Cok}_q \to H_q(X_k^u; C) \to \operatorname{Ker}_{q-1} \to 0.$$

So $\dim_C H_q(X_k^u; C) = \dim_C \mathrm{Ker}_{q-1} + \dim_C \mathrm{Cok}_q$.

Note that the homomorphism $(t^k - 1)$ respects any Ψ-splitting for $H_q(\widetilde{X}; C)$. It will therefore suffice in most instances to calculate the kernel and cokernel of $(t^k - 1)$ restricted to a single cyclic summand of the form Ψ/λ. λ has a prime factorization in Ψ into powers of linear factors $\lambda = \Pi_j (t - z_j)^{e_j}$. We can further decompose Ψ/λ into the direct sum

$$\Psi/(t - z_1)^{e_1} \oplus \cdots,$$

so focus attention on a prime-power summand $\Psi/(t - z_j)^{e_j}$. Since $(t^k - 1) = \Pi_{i=0}^{k-1} (t - \zeta^i)$ where ζ is a primitive kth root of unity, we have the exact sequence

$$0 \to \mathrm{Ker} \to \Psi/(t - z_j)^{e_j} \xrightarrow{(t^k - 1)} \Psi/(t - z_j)^{e_j} \to \mathrm{Cok} \to 0$$

where

$$\begin{aligned} \mathrm{Ker} \cong \mathrm{Cok} &\cong C \quad && \text{if } z_j \text{ is a } k\text{th root of unity,} \\ &\cong 0 && \text{otherwise.} \end{aligned}$$

Hence the theorem is proved in the range $3 \le q \le n$. Since

$$H_{n+2}(\widetilde{X}; C) \cong (\mu - 1)\Psi \quad \text{and} \quad 0 \to \Psi \xrightarrow{(t^k - 1)} \Psi - kC - 0$$

is exact, then $\mathrm{Cok}_{n+1} \cong k(\mu - 1)C$. So $\mathrm{rank}_C\, H_{n+1}(X_k^u; C) = k(\mu - 1) + c_n$. Likewise Ker_1 gets contributions only from the Ψ-torsion part of $H_1(\widetilde{X}; C)$ so $\mathrm{rank}_C\, H_2(X_k^u; C) = c_2 + c_1$. In dimension 1, we have the following exact sequence:

$$0 \to [k(\mu - 1) + c_1]C \to H_1(X_k^u; C) \to H_0(\widetilde{X}; C) \xrightarrow{0} H_0(\widetilde{X}; C) \to \cdots.$$

Hence $\mathrm{rank}_C\, H_1(X_k^u; C) = k(\mu - 1) + c_1 + 1$. This completes the proof of Theorem 1.

Corollary 2. *Let X_k^b be the k-fold branched cyclic cover of X, and $n \ge 2$ or $n = 1$, $\mu = 1$, and $\overline{\beta}_q$ = q-dimensional betti number of X_k^b. Then*

$$\begin{aligned} \beta_q &= c_q + c_{q-1}, && 2 \le q \le n, \\ &= (k - 1)(\mu - 1) + c_1, && q = 1, \\ &= (k - 1)(\mu - 1) + c_n, && q = n + 1, \\ &= 1, && q = 0,\ n + 2. \end{aligned}$$

Proof. By excision,

$$H_i(X_k^b, X_k^u; C) = \mu C, \quad i = 2, n+2,$$
$$= 0, \quad \text{otherwise.}$$

Just as in [5], we have that X_k^b is orientable, and the exact sequence of the pair (X_k^b, X_k^u) yields

$$0 \to H_{n+2}(X_k^b) \xrightarrow{j_*} H_{n+2}(X_k^b, X_k^u) \to H_{n+1}(X_k^u) \to H_{n+1}(X_k^b) \to 0.$$

Moreover, j_* takes the orientation class onto the element $(1, 1, \cdots, 1)$ in $C \oplus \cdots \oplus C \cong H_{n+2}(X_k^b; X_k^u)$, so we have that

$$\begin{aligned} \operatorname{rank}_C H_{n+1}(X_k^b; C) &= \operatorname{rank}_C H_{n+1}(X_k^u; C) - (\mu - 1) \\ &= k(\mu - 1) + c_n - (\mu - 1) = (k-1)(\mu - 1) + c_n . \end{aligned}$$

Likewise, it is shown in [5] that the following is a short exact sequence:

$$0 \to H_2(X_k^b, X_k^u) \xrightarrow{\partial} H_1(X_k^u) \to H_1(X_k^b) \to 0.$$

Hence $\operatorname{rank}_C H_1(X_k^b) = \operatorname{rank}_C (X_k^u) - \mu = k(\mu - 1) + c_1 + 1 - \mu = (k-1)(\mu-1) + c_1$. Otherwise, $H_q(X_k^b) \cong H_q(X_k^u)$, $2 \leq q \leq n$.

The previous analysis took place over C, and the algebraic fact that all the polynomial invariants factored into linear factors in Ψ considerably simplified the calculation. It is possible to make a similar analysis in Γ, which yields considerably more information in the case of prime power coverings. As before, $\overline{\beta}_q$ is the q-dimensional betti number of X_k^b.

Theorem 3. *$n \geq 2$ or $n = 1$, $\mu = 1$ and $k = p^r$, p a prime. Then*

$$\overline{\beta}_q = (k-1)(\mu - 1), \quad q = 1, n+1,$$
$$= 1, \quad q = 0, n+2,$$
$$= 0, \quad \textit{otherwise.}$$

Proof. Following the proof of Theorem 1, the analysis over Γ boils down to looking at $(t^k - 1)$ restricted to a cyclic summand Γ/λ. For the prime number p, we have the prime factorization in Γ:

$$(t^p - 1) = (t-1)(t^{p-1} + t^{p-2} + \cdots + 1)$$

where the second factor is the pth cyclotomic polynomial $\phi_p(t)$. Moreover, we have the prime factorization [6, p. 115] $(t^{p^r}) = (t-1)\phi_p(t)\phi_{p^2}(t) \cdots \phi_{p^r}(t)$

where $\phi_{p^\nu}(t) = 1 + t^{p^{\nu-1}} + t^{2p^{\nu-1}} + \cdots + t^{(p-1)p^{\nu-1}}$. Hence for $\nu \geq 1$ we have $\phi_{p^\nu}(1) \geq 2$. Now the rational polynomial invariants $\{\lambda_i^q(t)\}$, $1 \leq q \leq n$, have the property [5, Theorem 2.4] that $\lambda_i^q(1) = \pm 1$. This means that $\lambda_i^q(t)$ and $(t^{p^r} - 1)$ are relatively prime, so the contribution to $H_q(X_k^u; Q)$ from the Γ-torsion part of $H_q(\widetilde{X}; Q)$ is trivial for $q \geq 1$.

We are left with the case of a classical link, $n = 1$, $\mu \geq 2$. In this case, we have $H_2(\widetilde{X}; Q) = r_2\Gamma$, and

$$H_1(\widetilde{X}; Q) = r_1\Gamma \oplus \Gamma/\lambda_1^1 \oplus \cdots \oplus \Gamma/\lambda_{m_1}^1 .$$

Let ξ be the greatest integer such that $\lambda_\xi^1(1) = 0$. If $\lambda_1^1(1) \neq 0$, take $\xi = 0$.

Theorem 4. $n = 1$, $\mu \geq 2$. *If* $\overline{\beta}_q$ *is the* q*-dimensional betti number of* X_k^b, *then* $\overline{\beta}_1 = \overline{\beta}_2 = (k-1)(\mu - 1) + c_1 - k\xi$.

Proof. $\overline{\beta}_1 = \overline{\beta}_2$ by Poincaré duality, so it will suffice to compute $\overline{\beta}_2$. As in [4], [5] we have the short exact sequence of chain complexes

$$0 \to C_*(\widetilde{X}; Q) \xrightarrow{(t-1)} C_*(\widetilde{X}; Q) \to C_*(X; Q) \to 0$$

which induces the long exact sequence of homology

$$(3)\qquad \begin{array}{ccccccccc} 0 \to & H_2(\widetilde{X}; Q) & \xrightarrow{(t-1)} & H_2(\widetilde{X}; Q) \to & H_2(X; Q) & \to H_1(\widetilde{X}; Q) \to \cdots \\ & \| & & & \| & \\ & r_2\Gamma & & & (\mu-1)Q & \end{array}$$

If X fibers over S^1, then (3) is the Wang homology sequence of the fibration. From (3) we obtain the short exact sequence

$$0 \to r_2Q \to (\mu - 1)Q \to \xi Q \to 0 \quad \text{so } r_2 = (\mu - 1) - \xi .$$

Hence in the notation used in the proof of Theorem 1, we have

$$\mathrm{Cok}_2 = kr_2Q = k[(\mu - 1) - \xi]\,Q, \quad \text{and} \quad \mathrm{Ker}_1 = c_1Q.$$

So

$$\mathrm{rank}_Q(H_2(X_k^u; Q)) = k(\mu - 1) - k\xi + c_1,$$

and

$$\mathrm{rank}_Q(H_2(X_k^b; Q)) = \mathrm{rank}_Q(H_2(X_k^u; Q)) - (\mu - 1) = (k-1)(\mu - 1) + c_1 - k\xi .$$

This is the same as the result obtained in [3, Theorem 2], except that the present version gives an explicit calculation for $\overline{\beta}_1$ in terms of the Γ-torsion invariants.

As in Theorem 3, by considering prime-power coverings we obtain more information. The following corollary generalizes the corresponding result found by Durfee (see [1]) for 2-fold branched cyclic coverings of algebraic links, and in it we adopt the hypotheses and notation of Theorem 4:

Corollary 5. $n = 1$, $u \geq 2$ *and* $k = p^r$, p *a prime. Then* $\overline{\beta}_1 \leq (k-1)(\mu-1)$.

Proof. By the methods of [5, Theorem 2.4] we have

$$\lambda_i^1(1) = 0, \qquad i \leq \xi,$$
$$\qquad = \pm 1, \qquad i \geq \xi + 1.$$

Hence for k a prime power, this means that $(t^k - 1)$ and $\lambda_j^1(t)$ are relatively prime in Γ in the range $j \geq (\xi + 1)$. So the contribution to c_1 can come only from the $\lambda_j^1(t)$ with $j \leq \xi$, hence $c_1 \leq k\xi$.

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