COEFFICIENT DIFFERENCES AND HANKEL DETERMINANTS OF AREALLY MEAN p -VALENT FUNCTIONS

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ABSTRACT. With $p > 0$, denote by S_p the class of functions analytic and areally mean p -valent in the open unit disc. If $f \in S_p$, it is well known that $\alpha(f) = \lim_{r \rightarrow 1} (1-r)^{2p} M(r, f)$ exists and is finite. If $q \geq 1$ is an integer, denote the q th Hankel determinant of f by $H_q(n, f)$. In this paper the asymptotic behavior of $H_q(n, f)$, as $n \rightarrow \infty$, is related to $\alpha(f)$. A typical result is: if $\alpha(f) > 0$, and if $p > q - \frac{3}{4}$, then

$$|H_q(n, f)|/n^{2pq-q^2} \sim |Q_q(p)|(\alpha(f)/\Gamma(2p))^q,$$

where Q_q is a polynomial of degree at most $q^2 - q$. In the course of the proof, asymptotic results are proved concerning certain coefficient differences, and in particular concerning $|a_n| - |a_{n-1}|$.

1. **Introduction.** A function f analytic in $\gamma = \{z : |z| < 1\}$ is said to be areally mean p -valent in γ [5] if for all $R > 0$

$$\iint_{E(R)} |f'(z)|^2 \rho \, d\rho \, d\theta \leq p\pi R^2,$$

where $z = \rho e^{i\theta}$ and $E(R) = \{z \in \gamma : |f(z)| < R\}$. Throughout this paper we assume that f is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where f is normalized so that $\max_{1 \leq j \leq [p]} |a_j| = 1$. The class of normalized areally mean p -valent functions will be denoted by S_p .

The problem of estimating the rate of growth of combinations of coefficients of functions in S_p is well known (see [6, problem 6.14']). In addition to the well-known result of W. K. Hayman [3] that $||a_n| - |a_{n-1}|| = O(1)$ for $f \in S_1$, this problem has been studied in [1], [7], [8], [9], and [10], to men-

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tion only selected references. A closely related problem is the problem of estimating the rate of growth of the Hankel determinants for $f \in S_p$ (see [6, problem 6.13']). If $q \geq 1$ is an integer, the q th Hankel determinant of f is defined to be

$$H_q(n, f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Ch. Pommerenke has shown [9] that for $p \geq 1$, $H_q(n, f) = O(1)n^{k\sqrt{q}-q/2}$, with $k = 16p\sqrt{p}$. In particular, if q is large relative to p , $H_q(n, f) \rightarrow 0$ as $n \rightarrow \infty$. For strictly univalent functions, Pommerenke has shown [10] that for $q \geq 2$, $H_q(n, f) = O(1)n^{3/2-(\beta+1/2)q}$, where $\beta > 0$ is independent of f . In a recent paper with D. K. Thomas, the author showed [8] that $H_q(n, f) = O(1)n^{2pq-q^2}$ when $f \in S_p$ with $p \geq 2(q-1)$ and $q \geq 2$, and the exponent is best possible.

If $f \in S_p$, it is well known that $\alpha(f) = \lim_{r \rightarrow 1} (1-r)^{2p} M(r, f)$ exists and is finite, where as usual $M(r, f) = \max_{|z|=r} |f(z)|$ (see [1], [5]). It seems natural to expect that, at least when p is relatively large, the value of $\alpha(f)$ should determine the asymptotic behavior of both coefficient combinations and Hankel determinants of $f \in S_p$. This conjecture is supported by the following results (see [2], [3]); (i) $H_1(n, f) = a_n$, and $\lim_{n \rightarrow \infty} |a_n|/n^{2p-1} = \alpha(f)/\Gamma(2p)$ for $p > 1/4$; (ii) if $\alpha(f) > 0$, then $|a_n| - |a_{n-1}| \rightarrow \alpha(f)$ as $n \rightarrow \infty$ when $f \in S_1$. On the other hand, however, it is known [3] that (ii) need not hold if $\alpha(f) = 0$, and Hayman has also shown [4] that for $f \in S_1$, $H_2(n, f) = o(1)n^{1/2}$ and the exponent is best possible. In each case, $p = 1$ is "small", and $\alpha(f)$ does not determine the asymptotic behavior. In this paper we shall determine those values of p for which $\alpha(f)$ determines the asymptotic behavior of $H_q(n, f)$ and of certain natural coefficient combinations.

2. Statement of results. With j a nonnegative integer and x any complex number, put

$$\Delta_j(n, x, f) = \sum_{k=0}^j (-x)^k \binom{j}{k} a_{n+k},$$

so $\Delta_0(n, x, f) = a_n$, $\Delta_1(n, x, f) = a_n - xa_{n+1}$, $\Delta_2(n, x, f) = a_n - 2xa_{n+1} + x^2a_{n+2}$, and so forth. It was shown in [8] that if $f \in S_p$ and $p \geq j \geq 1$, then

there exists $\{e^{i\theta_n}\}$ such that $\Delta_j(n, e^{i\theta_n}, f) = O(1)n^{2p-j-1}$, and the exponent is best possible.

Theorem 1. *Let j be a positive integer.*

(i) *If $p > j$ and $f \in S_p$ with $\alpha(f) = 0$, there exists $\{e^{i\theta_n}\}$ depending only on f such that $\Delta_j(n, e^{i\theta_n}, f) = o(1)n^{2p-j-1}$ as $n \rightarrow \infty$.*

(ii) *If $p \leq j$, there exists $f \in S_p$ with $\alpha(f) = 0$ such that for any sequence $\{e^{i\theta_n}\}$,*

$$\limsup_{n \rightarrow \infty} \frac{|\Delta_j(n, e^{i\theta_n}, f)|}{n^{2p-j-1}} > 0.$$

Note that with $j = 1$, it follows from Theorem 1 that $|a_n| - |a_{n-1}| = o(1)n^{2p-2}$ if $p > 1$ and $\alpha(f) = 0$, while if $0 < p \leq 1$, the asymptotic behavior of $|a_n| - |a_{n-1}|$ is not determined by the fact that $\alpha(f) = 0$. Thus Theorem 1 generalizes the above-mentioned results of Hayman on the difference $|a_n| - |a_{n-1}|$.

Theorem 2. *Let $q \geq 1$ be an integer. If $f \in S_p$ and $\alpha(f) = 0$, then*

$$H_q(n, f) = o(1) \begin{cases} n^{2p-1}, & q = 1, p > 1/4, \\ n^{2pq-q^2}, & q \geq 2, p > 2(q-1). \end{cases}$$

We next consider the case $\alpha(f) > 0$. It is well known ([1], [5]) that in this case there exists a unique β such that $\alpha(f) = \lim_{r \rightarrow 1} (1-r)^{2p} |f(re^{i\beta})|$. We shall employ this notation consistently throughout this paper, and we shall also use the following notations

$$f_\beta(z) = (1 - ze^{-i\beta})^{-2p}, \quad r_n = n/(n+1),$$

$$\alpha_n(f) = (1 - r_n)^{2p} f(r_n e^{i\beta}), \quad f_n(z) = \alpha_n(f) f_\beta(z).$$

In our next theorem we determine those values of p for which $\Delta_j(n, e^{i\beta}, f)$ and $\Delta_j(n, e^{i\beta}, f_n)$ are asymptotically equivalent.

Theorem 3. *If $j \geq 0$, $p > (2j+1)/4$, and $f \in S_p$ with $\alpha(f) > 0$, then as $n \rightarrow \infty$,*

$$(\Delta_j(n, e^{i\beta}, f) - \Delta_j(n, e^{i\beta}, f_n))/n^{2p-j-1} \rightarrow 0.$$

The restriction $p > (2j+1)/4$ is best possible in the sense that given $\epsilon > 0$,

there exists $p \in ((2j+1)/4 - \epsilon, (2j+1)/4)$ and $F \in S_p$ with $\alpha(F) > 0$ such that for any sequence $\{e^{i\theta_n}\}$,

$$\limsup_{n \rightarrow \infty} \frac{|\Delta_j(n, e^{i\theta_n}, F)|}{n^{2p-j-1}} = +\infty.$$

In [8] it was shown that with $f_0(z) = (1-z)^{-2p}$,

$$\lim_{n \rightarrow \infty} \frac{|\Delta_j(n, 1, f)|}{n^{2p-j-1}} = \left| \prod_{k=1}^j (k-2p) \right|,$$

and so the exponent $2p-j-1$ above is clearly best possible. It was also shown in [8] that for $q \geq 1$ and for all $p > 0$,

$$(1) \quad \lim_{n \rightarrow \infty} \frac{|H_q(n, f_0)|}{n^{2pq-q^2}} = \frac{|Q_q(p)|}{\Gamma(2p)^q},$$

where Q_q is a nonconstant polynomial of degree at most $q^2 - q$. Using Theorem 3, we prove

Theorem 4. Let $q \geq 1$ and $p > q - 3/4$. If $f \in S_p$ and $\alpha(f) = \lim_{r \rightarrow 1} (1-r)^{2p} |f(re^{i\beta})| > 0$, then

$$\lim_{n \rightarrow \infty} \frac{|H_q(n, f)|}{n^{2pq-q^2}} = \left(\frac{\alpha(f)}{\Gamma(2p)} \right)^q |Q_q(p)|,$$

where Q_q is as above. The restriction $p > q - 3/4$ is best possible in the sense that given $q \geq 2$ and $\epsilon > 0$, there exists $p \in (q - 3/4 - \epsilon, q - 3/4)$ and $F \in S_p$ with $\alpha(F) > 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{H_q(n, F)}{n^{2pq-q^2}} = \infty.$$

3. The case $\alpha(f) = 0$. The proof of Theorem 1 is essentially the same as the proof of Theorem 2 of [8], and hence will not be repeated. The proof in [8] was based on estimates for certain integrals in terms of powers of $M(r, f)$. The full strength of the hypothesis $p \geq j$ in [8] was used only in the proof of Lemma 7-a of [8], where an integral was estimated in terms of $M(r, f)^{2(p-j)/p}$. If we now assume $p > j$, the exponent is positive, and since $\alpha(f) = 0$ implies $M(r, f) = o(1)(1-r)^{-2p}$, our estimate in [8] holds with $o(1)$ replacing $O(1)$. Careful analysis shows that with $p > j$, all $O(1)$ terms in

[8] can be replaced by $o(1)$, and so Theorem 1 is proved. Statement (ii) of Theorem 1 follows immediately from equation (6.3) of [8]. Theorem 2 follows directly from Theorem 1 upon using the method of [8, p. 519].

4. The case $\alpha(f) > 0$. We use the major-minor arc technique of Hayman [5] (see also [1]). With $r_n, \alpha_n(f), f_n(z)$ as before and with $x_n = r_n e^{i\beta}, z_n = r_n e^{i\theta}$, we have

$$(2) \quad |\Delta_j(n, x_n, zf') - \Delta_j(n, x_n, zf'_n)| \leq A(p, j) \left\{ \int_{E_1} |z - x_n|^j |f'(z) - f'_n(z)| d\theta + \int_{E_2} |z - x_n|^j |f'(z) - f'_n(z)| d\theta \right\},$$

where (with C an arbitrary constant) $E_1 = \{\theta: 0 \leq |\theta - \beta| \leq C(1 - r_n)\}$, $E_2 = \{\theta: C(1 - r_n) \leq |\theta - \beta| \leq \pi\}$. From [1, Lemma 15] it follows that $f'(z) - f'_n(z) = o(1)f'_n(z) = o(1)n^{2p+1}$, where as $n \rightarrow \infty$, $o(1)$ is uniform for $\theta \in E_1$. Since $|z - x_n| = r_n |e^{i(\theta-\beta)} - 1| = O(1)|\theta - \beta| = O(1)n^{-1}$ for $\theta \in E_1$, we conclude that as $n \rightarrow \infty$,

$$(3) \quad \int_{E_1} |z - x_n|^j |f'(z) - f'_n(z)| d\theta = o(1)n^{2p-j}.$$

(Although the $O(1)$ terms above depend on C , note that $o(1) \rightarrow 0$ as $n \rightarrow \infty$, for any fixed C , and so we get $o(1)$ in (3).)

In order to estimate the second integral in (2), choose $\eta > 0$ such that $p > (2j+1)/4 + \eta j$. This is possible since $p > (2j+1)/4$. Set $a = (4j)^{-1} - \frac{1}{2} + \eta$, $b = 2j(1+a)/p = p^{-1}(\frac{1}{2} + j + 2j\eta)$. With $z = r_n e^{i\theta}$, we apply the Cauchy-Schwarz inequality and find that

$$(4) \quad \begin{aligned} & \int_{E_2} |z - x_n|^j |f'(z)| d\theta \\ & \leq \left\{ \int_{E_2} |z - x_n|^{2j} |f(z)|^b d\theta \right\}^{1/2} \left\{ \int_{E_2} |f'(z)|^2 |f(z)|^{-b} d\theta \right\}^{1/2} \\ & = I_1^{1/2} I_2^{1/2} \quad \text{say.} \end{aligned}$$

With $\lambda = 2 - b$, we have that $I_2 \leq \int_0^{2\pi} |f'(z)|^2 |f(z)|^{\lambda-2} d\theta$. Our choice of η guarantees that $0 < \lambda \leq 2$, and so from [5, Lemma 3.1]

$$(5) \quad I_2 \leq A(p, \lambda, f) M(r_n, f)^\lambda / (1 - r_n) = O(1)n^{2p\lambda+1}.$$

We now estimate I_1 . Choose $\epsilon \in (0, 2p)$ so small that $2j\eta(2 - \epsilon/p) - (j + \frac{1}{2})\epsilon/p > 0$. By [5, Theorem 2.9] there exists $C > 0$ and $r_0 < 1$ such that

$|f(re^{i\theta})| \leq (1-r)^{-\epsilon} |\theta - \beta|^{\epsilon-2p}$ for $r_0 \leq r < 1$, $\theta \in \{\theta : C(1-r) \leq |\theta - \beta| \leq \pi\}$. Using the fact that $|z - x_n|^{2j} = O(1)|\theta - \beta|^{2j}$ and noting that $2pb - \epsilon b - 2j > 1$ by our choice of ϵ , we find that

$$I_1 \leq O(1)(1-r_n)^{-\epsilon} \int_{C(1-r_n)} |\theta - \beta|^{\epsilon b + 2j - 2pb} d\theta = O(1) \frac{n^{2pb - 2j - 1}}{C^{2pb - \epsilon b - 2j - 1}}.$$

Since $2pb - \epsilon b - 2j - 1 > 0$, and since C can be chosen as large as desired, it follows that given $\delta > 0$ there exists $C(\delta)$ and $r_0(\delta)$ such that

$$(6) \quad I_1 < \delta n^{2pb - 2j - 1}.$$

Combining (4), (5), (6), and noting that $b + \lambda = 2$, we see that

$$(7) \quad \int_{E_2} |z - x_n|^j |f'(z)| d\theta < \delta n^{2p-j}$$

for n sufficiently large. Clearly the same estimate holds for f'_n . Since $\delta > 0$ is arbitrary and since $o(1)$ in (3) approaches zero once δ (and hence $C(\delta)$) has been fixed, we conclude from (2), (3), and (6) that

$$(8) \quad (\Delta_j(n, x_n, zf') - \Delta_j(n, x_n, zf'_n))/n^{2p-j} = o(1).$$

The first statement of Theorem 3, which involves f and f_n as opposed to f' and f'_n , now follows by a simple induction argument from (8) and Lemma 2 of [8].

To prove the first statement of Theorem 4, we put $x = e^{i\beta}$ and expand $H_q(n, f)$, $H_q(n, f_n)$ by the symmetric group as in [8, p. 519]. We find that

$$(9) \quad H_q(n, f) - H_q(n, f_n) = \sum_{\mu} \left\{ \prod_{m=1}^q \Delta_{\mu(m)}(f) - \prod_{m=1}^q \Delta_{\mu(m)}(f_n) \right\},$$

where μ is a permutation with $0 \leq \mu(m) \leq 2q - 2$, $\sum_1^q \mu(m) = q^2 - q$. Now $p > q - \frac{3}{4}$ implies $p > (2\mu(m) + 1)/4$, so from Theorem 3,

$$\Delta_{\mu(m)}(f) = \Delta_{\mu(m)}(f_n) + o(1)n^{2p - \mu(m) - 1}.$$

It is easily verified that $\Delta_{\mu(m)}(f_n) = O(1)n^{2p - \mu(m) - 1}$, and so

$$(10) \quad \prod_{m=1}^q \Delta_{\mu(m)}(f) = \prod_{m=1}^q \Delta_{\mu(m)}(f_n) + o(1)n^{2pq - q^2}.$$

Thus from (9) and (10) we see that

$$(11) \quad H_q(n, f)/n^{2pq - q^2} = H_q(n, f_n)/n^{2pq - q^2} + o(1).$$

Simple computations show that

$$H_q(n, f_n) = \alpha_n(f)^q \exp(-i(n+q-1)q\beta) H_q(n, f_0),$$

where $f_0(z) = (1-z)^{-2p}$. Since $|\alpha_n(f)| \rightarrow \alpha(f)$ as $n \rightarrow \infty$, the first statement of Theorem 4 follows immediately from (1) and (11).

In order to show that the restrictions $p > (2j+1)/4$ and $p > q - 3/4$ of Theorems 3 and 4, respectively, are best possible, let $\{b_n\}$ be any sequence of nonnegative numbers satisfying $b_1 = 0$, $\sum_1^\infty b_n \leq 1$, $\sum_2^\infty nb_n^2 \leq p$. Set $\xi(z) = \sum_1^\infty b_n z^n$ and $F(z) = 2^{2p}\pi(1-z)^{-2p} + \xi(z)$. By making obvious modifications in the argument given by Hayman [4, Lemma 9], we see that for $p \geq 1/2$, $F \in S_p$. (For notational ease we neglect the normalization constant.) It is clear that $\alpha(F) = \lim_{r \rightarrow 1} (1-r)^{2p} |F(r)| > 0$, no matter what the sequence $\{b_n\}$.

To prove that the restriction $p > (2j+1)/4$ in Theorem 3 is best possible, choose $\epsilon > 0$ very small, set $p = (2j+1)/4 - \epsilon$, and choose a sequence $\{\delta_n\}$ such that $\lim_{n \rightarrow \infty} \delta_n = 0$, $\limsup_{n \rightarrow \infty} \delta_n n^{2\epsilon} = +\infty$. Now choose $\{b_n\}$ subject to the previous restrictions such that: (i) $b_n = \delta_n n^{-1/2}$ for infinitely many n ; (ii) $\lim_{n \rightarrow \infty} \delta_n n^{2\epsilon} = \infty$ for those integers chosen in (i); (iii) $b_n = 0$ except for the integers chosen in (i); (iv) if $b_n \neq 0$, then $b_{n+1} = \dots = b_{n+j} = 0$. (A specific example is as follows: choose an integer c such that $c > 1/\epsilon$, $m^c - (m-1)^c > j$ for all $m \geq 2$. If $k = m^c$ for some m , put $b_k = m^{-2/k^{1/2}} = m^{-2-c/2}$, and put $b_k = 0$ otherwise.)

Now let $\{e^{i\theta_n}\}$ be given. If for some subsequence $\{n_k\}$ we have that $\lim_{k \rightarrow \infty} \exp(i\theta_{n_k}) = a \neq 1$, then as in the proof of [8, Theorem 4] we have that $\limsup_{n \rightarrow \infty} \Delta_j(n, e^{i\theta_n}, G)/n^{2p-j-1} = \infty$, where $G(z) = 2^{2p}\pi(1-z)^{-2p}$. By using this fact in conjunction with the fact that

$$\Delta_j(n, e^{i\theta_n}, F) = \Delta_j(n, e^{i\theta_n}, G) + (-e^{i\theta_n})^k \binom{j}{k} b_{n+k}$$

for some k , $0 \leq k \leq j$ (this holds because of (iv) above), we conclude that $\limsup_{n \rightarrow \infty} \Delta_j(n, e^{i\theta_n}, F)/n^{2p-j-1} = \infty$, as required. If $\lim_{n \rightarrow \infty} e^{i\theta_n} = 1$, choose a subsequence (which we denote by $\{e^{i\theta_n}\}$) such that

$$\psi = \lim_{n \rightarrow \infty} n(1 - e^{i\theta_n})$$

exists and such that

$$(12) \quad \limsup_{n \rightarrow \infty} \frac{|\Delta_j(n, e^{i\theta_n}, G)|}{n^{2p-j-1}} = \lim_{n \rightarrow \infty} \frac{|\Delta_j(n, e^{i\theta_n}, G)|}{n^{2p-j-1}},$$

where $G(z) = 2^{2p}\pi(1-z)^{-2p}$, and where it is understood that the $\limsup$ in (12) is taken over the full sequence, while the limit is taken over the subsequence. Since $e^{i\theta_n} \rightarrow 1$, ψ is purely imaginary, with $\pm i\infty$ as possible values. If $\psi = \pm i\infty$, equations (6.6) and (6.7) of [8] imply that

$$\limsup_{n \rightarrow \infty} |\Delta_j(n, e^{i\theta_n}, F)|/n^{2p-j-1} = \infty.$$

If ψ is finite, it follows from (6.6) of [8] that the quantity in (12) is finite. We now choose another subsequence (again denoted by $\{e^{i\theta_n}\}$) such that $b_n = \delta_n n^{-1/2}$, $\delta_n n^{2\epsilon} \rightarrow \infty$. Condition (iv) above guarantees that $\Delta_j(n, e^{i\theta_n}, F) = \Delta_j(n, e^{i\theta_n}, G) + b_n$, and since $b_n n^{-2p+j+1} = \delta_n n^{2\epsilon}$, we deduce that $\limsup_{n \rightarrow \infty} \Delta_j(n, e^{i\theta_n}, F)/n^{2p-j-1} = \infty$ as required.

The proof that the restriction $p > q - \frac{3}{4}$ is best possible is very similar. With F as before, set $p = q - \frac{3}{4} - \epsilon$. Choose $\{\delta_n\}$ such that $\lim_{n \rightarrow \infty} \delta_n = 0$, $\lim_{m \rightarrow \infty} n_m^{2\epsilon} \delta_{n_m} = +\infty$ for some subsequence $\{n_m\}$. If $k = n_m$ for some m , set $b_{k+2q-2} = \delta_k k^{-1/2}$. Otherwise set $b_k = 0$. Then with $G(z) = 2^{2p}\pi(1-z)^{-2p}$, we have that

$$H_q(n_m, F) = H_q(n_m, G) + (-1)^{q+1} H_{q-1}(n_m, G) b_{n_m+2q-2}.$$

Upon combining this expression with (1), it follows that

$$\limsup_{m \rightarrow \infty} \frac{H_q(n_m, F)}{n_m^{2p-q-2}} = \infty \quad \text{iff} \quad \limsup_{m \rightarrow \infty} \frac{b_{n_m+2q-2}}{n_m^r} = \infty,$$

where $r = 2p + 1 - 2q$. But $2p - 2q + 1 = -\frac{1}{2} - 2\epsilon$, and $b_{n_m+2q-2} = \delta_{n_m} n_m^{-1/2}$, so

$$\limsup_{m \rightarrow \infty} \frac{b_{n_m+2q-2}}{n_m^r} = \limsup_{m \rightarrow \infty} \delta_{n_m} n_m^{2\epsilon} = \infty.$$

This completes the proof of Theorem 4.

5. Conclusion. If $f \in S_p$, then there exists an annulus $A(\delta, f) = \{z : 1 - 2\delta < |z| < 1\}$ in which f does not vanish, and so [5, p. 103] for any $\lambda > 0$, $g(z) = f(z)^\lambda$ may be continued analytically in $A(\delta, f)$. There exists μ such that in $A(\delta, f)$, $q(z)/z^\mu = \sum_{-\infty}^{+\infty} a_n z^n$.

It appears to be an open question as to whether or not g is areally mean λp -valent in $A(\delta, f)$. In any case, the main results of this paper continue to hold for such a g , provided we replace p by $p\lambda$ throughout. For example, if $q \geq 1$, $\alpha(f) > 0$, and $p\lambda > q - \frac{3}{4}$, then

$$\lim_{n \rightarrow \infty} \frac{|H_q(n, g)|}{n^{2p\lambda q - q^2}} = \left(\frac{\alpha(f)}{\Gamma(2p\lambda)} \right)^q |\mathcal{Q}_q(p\lambda)|.$$

The proofs for arbitrary $\lambda > 0$ are straightforward generalizations of the proofs for $\lambda = 1$, although the details become somewhat more complicated. These proofs are left to the interested reader.

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