THE INVERSE LIMIT AND FIRST DERIVED FUNCTOR, A SHORT EXACT SEQUENCE

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ABSTRACT. This paper provides a new topological proof that a certain sequence involving the inverse limit and first derived functor is short exact.

1. Introduction. The purpose of this paper is to provide a direct topological proof that the following is a short exact sequence:

$$0 \rightarrow \text{proj } \lim^1 [SY^n, X] \rightarrow [Y, X] \rightarrow \text{proj } \lim [Y^n, X] \rightarrow 0.$$

For the purposes of this paper we will restrict ourselves to the category of pointed sets. The inverse system used above is that induced from the inclusion of the n -skeleton Y^n into Y^{n+1} and $\text{proj } \lim^1$ denotes the first derived functor. This fact is known and it or similar statements appear in a number of places in the literature. For example, Gray [3] states the result without proof, while Milnor [4] proves similar statements for homology and cohomology, providing the idea for the proof presented here. One could also arrive at this result as a simple application of Theorem 5.3 of Eckmann and Hilton [2]. However, there does not appear to be a direct proof in the literature.

All spaces will be connected and of the homotopy type of CW-complexes. All maps and homotopies will be basepoint preserving.

2. Preliminaries. The notion of a cokernel will be needed to define the first derived functor. Cokernels exist in any category that has zero maps. We work in the category of pointed sets, and cokernels exist in this category.

Definition 1. Let $\mathcal{C}$ be a category with a zero morphism and let X and Y be objects in $\mathcal{C}$. Let f be a morphism from X to Y . The *cokernel* of f is an object Z of $\mathcal{C}$ and a morphism q from Y to Z such that $q \circ f$ is the zero morphism and q and Z are universal with respect to this property.

$$\begin{array}{ccccc} X & \xrightarrow[f]{0} & Y & \xrightarrow{q} & Z \\ & & \searrow q' & & \downarrow h \\ & & & & Z' \end{array}$$

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Let $f: X \rightarrow Y$ in the category of pointed sets. The cokernel of f turns out to be $Y/f(X)$. That is, all points in Y of the form $f(x)$ are identified with the basepoint.

Definition 2. Let $\{G_n, p_n\}$ be an inverse system of pointed sets. Define the *inverse limit* of $\{G_n, p_n\}$, denoted $\text{proj lim } G_n$, as $\{\prod_n g_n \in \prod_n G_n : g_n = p_{n+1}(g_{n+1})\}$. If $\{G_n, p_n\}$ is an inverse system of groups, define $u: \prod_n G_n \rightarrow \prod_n G_n$ by $u(\prod_n g_n) = \prod_n (g_n \cdot p_{n+1}(g_{n+1})^{-1})$. The *first derived functor* of the inverse limit of $\{G_n, p_n\}$, denoted $\text{proj lim}^1 G_n$, is defined as the cokernel of u in the category of pointed sets.

We remark here that proj lim^1 even changes its underlying set if one passes to the category of abelian groups.

For our proof, we follow Milnor [4] and define $L = \bigcup_{n \geq 1} Y^n \times [n-1, n]/\sim$, where $\sim$ identifies $(x, n-1) \in Y^n \times [n-1, n]$ and $(x, n-1) \in Y^{n-1} \times [n-2, n-1]$ and $\sim$ also identifies $(*, i)$ and $(*, 0)$. The procedure is to construct a Mayer-Vietoris type argument using the excisive couple (L_1, L_2) where $L_1 = \bigcup_{n \text{ odd}} Y^n \times [n-1, n]/\sim$ and $L_2 = \bigcup_{n \text{ even}} Y^n \times [n-1, n]/\sim$ are subspaces of L . Note that $h: L \rightarrow Y$ by $h(y, i) = y$ is a homotopy equivalence. Pictorially L is:

L_1 is the shaded portion, L_2 is the unshaded portion, and $L_1 \cap L_2$ is the darkened portion. Since $L \simeq Y$, $L_1 \simeq \bigvee_{n \text{ odd}} Y^n$, $L_2 \simeq \bigvee_{n \text{ even}} Y^n$, and $L_1 \cap L_2 \simeq \bigvee_{n \geq 1} Y^n$, the following is obvious:

- Proposition 3.** (1) $[L_1 \cap L_2, X] \approx \prod_n [Y^n, X]$;
 (2) $[L_1, X] \approx \prod_{n \text{ odd}} [Y^n, X]$;
 (3) $[L_2, X] \approx \prod_{n \text{ even}} [Y^n, X]$;
 (4) $[L, X] \approx [Y, X]$.

We need the dual version of Barratt and Whitehead's Lemma 7.4 [1] for a homotopy version of Mayer-Vietoris.

Lemma 4. Suppose that the following is a commutative diagram of groups and homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
 G_0 & \xleftarrow{\lambda_0} & G_1 & \xleftarrow{\lambda_1} & G_2 & \xleftarrow{\lambda_2} & G_3 & \xleftarrow{\lambda_3} & G_4 & \xleftarrow{\lambda_4} & G_5 & \xleftarrow{\dots} \\
 \downarrow \theta_0 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 & & \downarrow \theta_4 & & \downarrow \theta_5 & \\
 H_0 & \xleftarrow{\mu_0} & H_1 & \xleftarrow{\mu_1} & H_2 & \xleftarrow{\mu_2} & H_3 & \xleftarrow{\mu_3} & H_4 & \xleftarrow{\mu_4} & H_5 & \xleftarrow{\dots}
 \end{array}$$

Suppose further that θ_0 is monic and that θ_{3k} is an isomorphism for all $k = 1, 2, 3, \dots$. Then,

$$\begin{array}{c}
 G_1 \xleftarrow{u} G_2 \times H_1 \xleftarrow{v} H_2 \xleftarrow{w} G_4 \xleftarrow{\dots} G_i \\
 \qquad \qquad \qquad \xleftarrow{u} G_{i+1} \times H_i \xleftarrow{v} H_{i+1} \xleftarrow{w} G_{i+3} \xleftarrow{\dots}
 \end{array}$$

is exact where $i \equiv 1 \pmod{3}$, $u = \lambda_i \cdot \theta_i^{-1}$, $v = (\theta_{i+1}, \mu_i)$, and $w = \mu_{i+1} \circ \theta_{i+2}^{-1} \circ \lambda_{i+2}$. Moreover, if G_i and H_i are just pointed sets for $i = 1, 2$, and 3 (and need not exist for $i = 0$), then the above sequence is still exact if terminated at $G_2 \times H_1 \xleftarrow{v} H_2 \xleftarrow{w} \dots$.

Proof. Since the diagram commutes and its rows are exact, image $v \subset \text{kernel } u$, image $w \subset \text{kernel } v$, and image $u \subset \text{kernel } w$. The reverse inclusions are all simple diagram chases. Note that if the groups fail to be abelian, u need not be a homomorphism. Q.E.D.

Consider $L_1 \cap L_2 \hookrightarrow L_1 \rightarrow L_1 \cup c(L_1 \cap L_2)$ and $L_2 \hookrightarrow L = L_1 \cup L_2 / \sim \rightarrow L \cup c(L_2)$. Then $L_1 \cup c(L_1 \cap L_2) \cong L \cup c(L_2)$ since (L_1, L_2) is an excisive couple. The Puppe sequences

$$L_1 \cap L_2 \xrightarrow{i} L_1 \xrightarrow{j} L_1 \cup c(L_1 \cap L_2) \xrightarrow{\delta} S(L_1 \cap L_2) \xrightarrow{Si} SL_1 \rightarrow \dots$$

and

$$L_2 \xrightarrow{i'} L \xrightarrow{j'} L \cup c(L_2) \xrightarrow{\delta'} SL_2 \xrightarrow{Si'} SL \rightarrow \dots$$

yield the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 [L_1 \cap L_2, X] & \xleftarrow{i^* \equiv \lambda_1} & [L_1, X] & \xleftarrow{j^* \equiv \lambda_2} & [L_1 \cup c(L_1 \cap L_2), X] & \xleftarrow{\delta^* \equiv \lambda_3} & \dots \\
 \uparrow \theta_1 & & \uparrow \theta_2 & & \approx \uparrow \theta_3 & & \\
 [L_1, X] & \xleftarrow{i'^* \equiv \mu_1} & [L, X] & \xleftarrow{j'^* \equiv \mu_2} & [L \cup c(L_2), X] & \xleftarrow{\delta'^* \equiv \mu_3} & \dots
 \end{array}$$

The θ_i are induced by the appropriate inclusion maps and θ_{3k} is an isomorphism for $k = 1, 2, \dots$. This is the situation required for Lemma 4.

3. The exact sequence.

Theorem 5.

$$0 \rightarrow \text{proj lim}^1 [SY^n, X] \xrightarrow{w} [Y, X] \xrightarrow{v} \text{proj lim} [Y^n, X] \rightarrow 0$$

is exact as a sequence of pointed sets, and is natural in the second factor.

Moreover, if either Y is a co- H -group or X is a homotopy associative H -space, then whenever $f, g: Y \rightarrow X$ are such that $v([f]) = v([g])$, $[f] \circ [g]^{-1} \in \text{image } w$.

(We should remark that the moreover part of the theorem merely needs a natural group structure on $[Y, X]$.)

Proof. The above permits the application of Lemma 4 to conclude that

$$\begin{aligned} [L_1, X] \times [L_2, X] &\xleftarrow{v} [L, X] \xleftarrow{w} [S(L_1 \cap L_2), X] \\ &\xleftarrow{u} [SL_1, X] \times [SL_2, X] \leftarrow \dots \end{aligned}$$

is exact. Remembering what L_1, L_2, L , and $L_1 \cap L_2$ are yields via Proposition 3 the exact sequence

$$(*) \quad \prod_n [Y^n, X] \xleftarrow{v} [Y, X] \xleftarrow{w} \prod_n [SY^n, X] \xleftarrow{u} \prod_n [SY^n, X] \leftarrow \dots$$

Recall that $u: \prod_n [SY^n, X] \rightarrow \prod_n [SY^n, X]$ comes from

$$u: [SL_1, X] \times [SL_2, X] \rightarrow [S(L_1 \cap L_2), X]$$

where $u([g], [h]) = \lambda_4([g]) \cdot \theta_4([h]^{-1})$. Hence, if $\prod_n [g_n] \in \prod_n [SY^n, X]$, then $u(\prod_n [g_n]) = \prod_n ([g_n] \cdot [g_{n+1} \circ l]^{-1})$ where l is simply the inclusion of the skeletons, e.g., $SY^n \rightarrow SY^{n+1}$. Thus, $\ker u = \text{proj lim} [SY^n, X]$ and $\text{coker } u = \text{proj lim}^1 [SY^n, X]$. Hence, $0 \rightarrow \text{proj lim}^1 [SY^n, X] \xrightarrow{w} [Y, X] \xrightarrow{v} \text{image } v \rightarrow 0$ is exact as pointed sets. The basepoint of $[Y, X]$ is the class of the constant map to the basepoint of X , the basepoint of $\text{image } v$ is the product of the classes of the appropriate constant maps, and the basepoint of $\text{proj lim}^1 [SY^n, X]$ is the class of the product of the classes of the appropriate constant maps.

To complete the first half of the proof we merely have to identify $\text{image } v$ with $\text{proj lim} [Y^n, X]$. Recall that $\text{image } v \subset [L_1, X] \times [L_2, X]$ consists of all $([g], [h])$ where $g: L_1 \rightarrow X, h: L_2 \rightarrow X$ and $([g], [h]) = v([d])$ for some $d: Y \rightarrow X$. Now, $v([d]) = (\mu_1([d]), \theta_2([d])) = ([d \circ i'], [d \circ k])$, where $i': L_2 \rightarrow L$ and $k: L_1 \rightarrow L$ are the inclusions. Thus, using the identifications of Proposition 3, $v([d]) = ([g], [h]) \in [L_1, X] \times [L_2, X]$ gets identified to $\prod_n [d|Y^n] \in \prod_n [Y^n, X]$. This is certainly in $\text{proj lim} [Y^n, X]$ since $(d|Y^n)|Y^{n-1} \simeq d|Y^{n-1}$. Hence, $\text{image } v \subset \text{proj lim} [Y^n, X]$. For the other direction, suppose

that $\Pi_n[f_n] \in \text{proj lim } [Y^n, X] \subset \Pi_n[Y^n, X]$. Then $f_n|Y^{n-1} \simeq f_{n-1}$ by some homotopy $H'_{n-1}: Y^{n-1} \times I \rightarrow X$. This induces $H_{n-1}: Y^{n-1} \times I/\{(*, t)\} \rightarrow X$. Define $f: L \rightarrow X$ by $f(y, t) = H_n(y, t)$ if $n-1 \leq t \leq n$. Pictorially:

Now, $f|Y^i \times \{i\} = f_{i+1}|Y^i \simeq f_i$. Hence, $v([f]) = \Pi_n[f_n]$, showing that image $v = \text{proj lim } [Y^n, X]$. Since naturality is clear, the first half of the proof is now complete.

For the second statement, assume that either Y is a co- H -group or that X is a homotopy associative H -space. Then $[Y, X]$ and each $[Y^n, X]$ are groups [5, p. 10], while each $[SY^n, X]$ is an abelian group (if Y is a co- H -group, assume that the comultiplication is cellular so that Y^n is a co- H -group). Let $f, g: Y \rightarrow X$. In the sequence (*), all sets are now groups and all maps are homomorphisms. Suppose that $v([f]) = v([g])$. Then

$$v([f] \cdot [g]^{-1}) = v([f]) \cdot v([g])^{-1} = 1.$$

Hence set exactness implies that there is an $a \in \text{proj lim}^1[SY^n, X]$ such that $w(a) = [f] \cdot [g]^{-1}$. Q.E.D.

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