ALTERNATING FORMS AND ONE-RELATOR GROUPS

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ABSTRACT. This paper considers epimorphisms of finitely generated one-relator groups $G = F/R$, F free, $R \subseteq F^p[F, F]$ for some rational prime p . The main result is a lower bound for the difference $\text{rank } G - \text{rank } G'$ when G' is a one-relator group homomorphic to G . This generalizes a known result for the case G' a free group.

Let V be a finite-dimensional vector space over a field F equipped with an alternate bilinear form $(X, Y) \mapsto XY \in F$; $X, Y \in V$ ($XX = 0$ for every $X \in V$). Given a subset S of V let S^* denote the subspace $\{X \in V \mid XY = 0 \text{ for all } Y \in S\}$. A subspace W of V is called *isotropic* if $XY = 0$ for all $X, Y \in W$. The *radical* V^* of V is denoted by $\text{rad } V$ and is clearly isotropic. If W is a subspace of V then $\text{rad } W = W \cap W^*$. W is *nonsingular* if $\text{rad } W = 0$.

Lemma 1. *Let V be a finite-dimensional space over a field F equipped with an alternate bilinear form $(X, Y) \mapsto XY \in F$, and let U be a subspace of V . Then U has a direct sum decomposition*

$$U = U_1 \oplus U_0 \oplus (\text{rad } U \cap \text{rad } V),$$

where $U_0 \oplus (\text{rad } U \cap \text{rad } V) = \text{rad } U$, and U_1 is nonsingular. Furthermore, this decomposition can be extended to a decomposition

$$V = V_1 \oplus U_1 \oplus U_0 \oplus U'_0 \oplus (\text{rad } U \cap \text{rad } V) \oplus V_0,$$

where U'_0 is an isotropic subspace of V of the same dimension as U_0 and $U_0 \oplus U'_0$ is nonsingular, V_1 is nonsingular, and $\text{rad } V = (\text{rad } U \cap \text{rad } V) \oplus V_0$.

Proof. By [1, Theorem 3.3, p. 116], $U = U_1 \oplus \text{rad } U$, U_1 nonsingular. Write $\text{rad } U = (\text{rad } U \cap \text{rad } V) \oplus U_0$, $\text{rad } V = (\text{rad } U \cap \text{rad } V) \oplus V_0$; U_0, V_0 subspaces of V . Since $U_0 \cap \text{rad } V = 0$, $U + \text{rad } V$ is the direct sum

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$U_1 \oplus U_0 \oplus \text{rad } V$. Write $V = V_1' \oplus U_1 \oplus U_0 \oplus \text{rad } V$. Then $V_2 = V_1' \oplus U_1 \oplus U_0$ is nonsingular. By [1, Theorem 3.5, p. 117], $V_2 = U_1 \oplus (U_1^* \cap V_2)$, and $U_1^* \cap V_2$ is nonsingular. Also $U_1^* \cap V_2 \supseteq U_0$. By [1, Theorem 3.8, p. 120] there exists an isotropic subspace U_0' of (the nonsingular space) $U_1^* \cap V_2$ such that $U_0 \oplus U_0'$ is nonsingular. Set $V_1 = (U_0 \oplus U_0')^* \cap (U_1^* \cap V_2)$. We may now write $U_1^* \cap V_2 = V_1 \oplus U_0 \oplus U_0'$, and finally, $V = V_2 \oplus \text{rad } V = V_1 \oplus U_0 \oplus U_0' \oplus U_1 \oplus (\text{rad } U \cap \text{rad } V) \oplus V_0$. Q.E.D.

Let F be the free group on n generators. Let p be an odd rational prime and define the p -central series recursively: $F_0 = F$, $F_m = F_{m-1}^p [F, F_{m-1}]$, where F_{m-1}^p , $[F, F_{m-1}]$ denote the subgroups generated by p th powers of elements of F_{m-1} , and the commutators of elements of F_{m-1} with elements of F , respectively. Let $G = F/R$, where R is the normal subgroup of F generated by an element $r \in F_1$. Let F' be the free group on $m \leq n$ generators, $G' = F'/R'$, R' an arbitrary normal subgroup of F' contained in F_1' (where $\{F_i'\}$ is defined analogous to $\{F_i\}$).

Suppose there is an epimorphism $f: G \rightarrow G'$. f induces a monomorphism $f^*: H^1(G') \rightarrow H^1(G)$ where $H^i(G)$, $H^i(G')$ denote the i th cohomology groups $H^i(G, \mathbb{Z}/p\mathbb{Z})$, $H^i(G', \mathbb{Z}/p\mathbb{Z})$, respectively, over $\mathbb{Z}/p\mathbb{Z}$, with trivial action on $\mathbb{Z}/p\mathbb{Z}$. We have a commutative diagram

$$(1) \quad \begin{array}{ccc} H^1(G') \times H^1(G') & \xrightarrow{\cup} & H^2(G') \\ \downarrow & & \downarrow \\ H^1(G) \times H^1(G) & \xrightarrow{\cup} & H^2(G) \end{array}$$

where $\cup$ denotes the cup product, and the vertical arrows are induced by f . One can show that $H^2(G) \cong \mathbb{Z}/p\mathbb{Z}$ so that the cup product makes $V = H^1(G)$ an n -dimensional $\mathbb{Z}/p\mathbb{Z}$ -space equipped with an alternate bilinear form. (For details, see [2].) We set $U = f^*H^1(G')$, which is a subspace of V .

It is shown in [2] that the mapping $\bar{r}: H^2(G) \rightarrow \mathbb{Z}/p\mathbb{Z}$ defined by $\bar{r}(a) = tg^{-1}(a)(r^{-1}) \in \mathbb{Z}/p\mathbb{Z}$, $a \in H^2(G)$, where $tg: H^1(R)^F \rightarrow H^2(G)$ is the transgression map, is an isomorphism. The bilinear form induced on V by the cup product will be defined to be $\bar{r}(\chi \cup \chi')$, $\chi, \chi' \in V$. We will henceforth write $\chi\chi' = \bar{r}(\chi \cup \chi')$.

We now apply Lemma 1 and obtain a direct sum decomposition

$$V = V_1 \oplus U_1 \oplus U_0 \oplus U_0' \oplus (\text{rad } U \cap \text{rad } V) \oplus V_0,$$

$$\text{where } (\text{rad } U \cap \text{rad } V) \oplus V_0 = \text{rad } V,$$

$$V_1, U_1, U_0 \oplus U_0' \text{ nonsingular, } U = U_1 \oplus \text{rad } U, \text{rad } U = U_0 \oplus (\text{rad } U \cap \text{rad } V),$$

U_0, U'_0 isotropic of the same rank. By [1, Theorem 3.7, p. 119] there is a basis $\alpha_1, \dots, \alpha_{2l}$ of V_1 such that $\alpha_{2i-1}\alpha_{2i} = 1$ for $1 \leq i \leq l$, $\alpha_i\alpha_j = 0$ for $i < j$ otherwise, $1 \leq i, j \leq 2l$.

Similarly, there is a basis $\beta_1, \dots, \beta_{2s}$ of U_1 such that $\beta_{2i-1}\beta_{2i} = 1$, etc. By [1, Theorem 3.8] there are bases $\gamma_1, \gamma_3, \dots, \gamma_{2t-1}$ of U_0 , and $\gamma_2, \gamma_4, \dots, \gamma_{2t}$ of U'_0 such that $\gamma_{2i-1}\gamma_{2i} = 1$, $1 \leq i \leq t$, $\gamma_i\gamma_j = 0$ otherwise, $1 \leq i < j \leq 2t$. Finally we take arbitrary bases $\delta_1, \dots, \delta_k$ of $\text{rad } U \cap \text{rad } V$ and $\rho_1, \dots, \rho_q$ of V_0 .

Let $\{\chi_1, \dots, \chi_n\}$ be the union of all the bases defined below. It is clearly a basis of V . Similarly, let $\eta_1, \dots, \eta_m$ be the basis of $H^1(G')$ such that $\{f^*(\eta_1), \dots, f^*(\eta_m)\}$ is the union of the bases of U_0, U_1 , and $\text{rad } U \cap \text{rad } V$ given below.

We may lift f to a homomorphism $f': F \rightarrow F'$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{f'} & F' \\ \downarrow e & & \downarrow e' \\ G & \xrightarrow{f} & G' \end{array}$$

commutes, where e, e' are the canonical mappings. For let $z_1, \dots, z_n$ be a basis of F . We may choose $u_1, \dots, u_n \in F'$ such that $fe(z_i) = e'(u_i)$, $1 \leq i \leq n$, and there is a unique homomorphism $f': F \rightarrow F'$ such that $f'(z_i) = u_i$, $1 \leq i \leq n$ (a basic property of free groups). There is induced a commutative diagram

$$\begin{array}{ccc} H^1(F) & \xleftarrow{f'^*} & H^1(F') \\ \uparrow e^* & & \uparrow e'^* \\ H^1(G) & \xleftarrow{f^*} & H^1(G') \end{array}$$

by means of which we identify $H^1(F)$ with $H^1(G)$, $H^1(F')$ with $H^1(G')$, and hence f^* with f'^* . (Note that $f'(F)F'_1 = F'$, hence, f'^* is monomorphic.)

We utilize these identifications in the following discussion.

By duality there exist $x_1, \dots, x_n \in F$ such that $\chi_i(x_j) = \delta_{ij}$, $1 \leq i, j \leq n$, and $y_1, \dots, y_m \in F'$ such that $\eta_i(y_j) = \delta_{ij}$, $1 \leq i, j \leq m$. These are not necessarily bases of F and F' ; however, by Burnside's basis theorem, they do map onto bases of F/F_2 and F'/F'_2 respectively.

Now for $1 \leq i \leq m$, $1 \leq j \leq n$, $\eta_i f'(x_j) = f'^*(\eta_i)(x_j) = \chi_{k(i)}(x_j) = \delta_{k(i), j}$ for some $k(i)$, $1 \leq k(i) \leq n$, where $\chi_{k(i)} = f^*(\eta_i) \in U$. If $j \notin \{k(i): 1 \leq i \leq m\}$,

then $\eta_i f'(x_j) = 0$ for $1 \leq i \leq m$, hence $f'(x_j) \equiv 1 \pmod{F'_1}$. On the other hand, if $j = k(t)$ for some t , $1 \leq t \leq m$, then $\eta_i f'(x_j) = \delta_{k(i), k(t)} = \delta_{it} = \eta_i(y_t)$ for $1 \leq i \leq m$, hence $f'(x_j) \equiv y_t \pmod{F'_1}$.

By [2, Proposition 1], the relator r satisfies the congruence

$$(2) \quad r \equiv \prod_{i=1}^n x_i^{p a_i} \prod_{i=1}^{l+s+t} [x_{2i-1}, x_{2i}] \pmod{F_2}$$

where $a_i \in \mathbb{Z}/p\mathbb{Z}$, and $\{\chi_1, \dots, \chi_{2l+2s+2t}\} = \{\alpha_1, \dots, \alpha_{2l}, \beta_1, \dots, \beta_{2s}, \gamma_1, \dots, \gamma_{2t}\}$ in that order. It follows that

$$f'(r) \equiv \prod_{i=1}^m y_i^{p b_i} \cdot \prod_{i=1}^s [y_{2i-1}, y_{2i}] \pmod{F'_2},$$

where $b_i = a_{k(i)}$, $\chi_{k(i)} = f^*(\eta_i)$, and $\{\beta_1, \dots, \beta_{2s}\} = \{f^*(\eta_1), \dots, f^*(\eta_{2s})\}$ in that order. We have therefore proved the following theorem.

Theorem 1. *Let G be a finitely generated one-relator group F/R , F free of rank n , R the normal subgroup of F generated by an element $r \in F_1 = F^p[F, F]$, p an odd rational prime. Suppose f is an epimorphism of G onto a group $G' = F'/R'$, where F' is free on $m \leq n$ generators and $R' \leq F'_1$. Then there exist $x_1, \dots, x_n \in F$, $y_1, \dots, y_m \in F'$, which map onto bases of F/F_2 , F'/F'_2 respectively ($F_2 = F^p[F_1, F]$), and nonnegative integers l, s, t, k, q , such that $2l + 2s + 2t + k + q = n$, $2s + t + k = m$, such that r takes the form (2) above $\pmod{F_2}$, and f lifts to a homomorphism $f': F \rightarrow F'$ such that*

- (i) f' maps $x_1, \dots, x_{2l}; x_{2(l+s+1)}, \dots, x_{2(l+s+t)}; x_{n-q+1}, \dots, x_n$ all into F'_1 ;
- (ii) f' maps $x_{2l+1}, \dots, x_{2l+2s}; x_{2(l+s)+1}, x_{2(l+s)+3}, \dots, x_{2(l+s+t)-1}; x_{2(l+s+t)+1}, \dots, x_{2(l+s+t)+k}$, onto $y_1, \dots, y_m \pmod{F'_2}$, in the given order.

There are two special cases of particular interest. The first is when G' is free or even when $R' \leq F'_2$. In this instance $s = 0$ since $f'(r) \in R'$, and for each pair $\{x_{2i-1}, x_{2i}\}$, $1 \leq i \leq l+t$, f' either maps both into F'_1 or x_{2i-1} onto a y_j and x_{2i} into F'_1 . The remaining x_i map either onto a y_j or into F'_1 . Furthermore, $m = t + k$ and $n = 2l + 2t + k + q$, hence $n - m = 2l + t + q \geq l + t$. Hence in general if G' is free, then

$$(3) \quad \text{rank } G' \leq \text{rank } G - \rho/2,$$

where $\rho = 2(l+t)$ is the dimension of $V/\text{rad } V$. This inequality is obtained in [2].

The second case of interest is when G' is a one-relator group, R' being the normal subgroup of F' generated by an element $r' \in F'_1$. Since $f'(r) \in R'$, $f'(r) \equiv r^b \pmod{F'_2}$ for some $b \in \mathbb{Z}$, $0 \leq b \leq p-1$. Furthermore,

$$\begin{aligned} n - m &= (2l + 2s + 2t + k + q) - (2s + t + k) = 2l + t + q \\ &\geq 2l + t \geq l + t = \frac{1}{2}(\rho(G) - \rho(G')), \end{aligned}$$

where $\rho(G) = \dim(V/\text{rad } V)$ and $\rho(G') = \dim(U/\text{rad } U)$. We thus have the following theorem.

Theorem 2. *Let $G = F/R$, $G' = F'/R'$ be (one-relator) presentations of one-relator groups G, G' such that $R \leq F_1$, $R' \leq F'_1$. If there is an epimorphism of G onto G' , then*

$$(4) \quad \text{rank } G - \text{rank } G' \geq \frac{1}{2}(\rho(G) - \rho(G')),$$

where $\rho(G) = \dim(H^1(G, \mathbb{Z}/p\mathbb{Z})/\text{rad } H^1(G, \mathbb{Z}/p\mathbb{Z}))$, $\rho(G')$ defined similarly, where $H^1(G, \mathbb{Z}/p\mathbb{Z})$, $H^1(G', \mathbb{Z}/p\mathbb{Z})$ are considered as $\mathbb{Z}/p\mathbb{Z}$ -spaces with an alternate bilinear form given by the cup product. (Note. We have utilized the commutativity of diagram (1) to obtain $\rho(G') = \dim(U/\text{rad } U)$.)

Remarks. 1. By the proof of Theorem 1, $\rho(G)$ can also be defined as the number of x_i occurring in the "canonical" form $\pmod{F_2}$ $\prod_{i=1}^{\alpha(G)/2} [x_{2i-1}, x_{2i}]$ of a relator r defining G .

2. Inequality (4) can be looked upon as a generalization of (3) since if G' is free, $\rho(G') = 0$.

3. Suppose $r \in [F, F]$, the commutator subgroup of F . Then $r \in F_1 = F^p[F, F]$ for every p . It is shown in [2] that there is a basis $x_1, \dots, x_n$ of F such that

$$r \equiv \prod_{i=1}^{n'} [x_{2i-1}, x_{2i}]^{a_i} \pmod{F_2^{(0)}},$$

where $F_2^{(0)} = [F, [F, F]]$ and $0 < a_i \in \mathbb{Z}$, $2n' \leq n$, and that $\rho_0 = \rho_0(G) = 2n'$ is the rank of the cup product $H^1(G, \mathbb{Z}) \times H^1(G, \mathbb{Z}) \rightarrow H^2(G, \mathbb{Z}) \simeq \mathbb{Z}$ as a $\mathbb{Z}$ -bilinear form. If we choose p so that p does not divide any of the a_i , then reducing $r \pmod{F_2} (\supset F_2^{(0)})$ yields $r \equiv \prod_{i=1}^{n'} [x_{2i-1}, x_{2i}^{a_i}] \pmod{F_2}$, a "canonical" form for $r \pmod{F_2}$, hence by Remark 1, $\rho(G) = \rho_0(G)$, where $\rho(G)$, as defined above, is the rank of the cup product on $H^1(G, \mathbb{Z}/p\mathbb{Z})$. Thus (3) yields an inequality due to A. Steinberg [2], and (4) the corresponding generalization of it.

4. Let f_1 be an epimorphism from F/RF_2 onto $F'/R'F'_2$, where F, R, F', R' are as in Theorem 1. f_1 lifts to an $f': F \rightarrow F'$ by the same argu-

ment used in lifting f to f' . By utilizing the mapping $f: F/R \rightarrow F'/R'F'_2$ induced by f' , and the proof of Theorem 1, we obtain a version of Theorem 1 with f replaced by f_1 . On the other hand, every f induces an f_1 , so the new version yields the original as well. Similarly, in Theorem 2, if we observe that the only assertions about f' that are made are about the composite of f' with the canonical mapping $F' \rightarrow F'/F'_2$, we obtain an equivalent version of Theorem 2 with f replaced by $f_1: F/RF_2 \rightarrow F'/R'F'_2$.

This last remark, together with Remark 3, yields (the apparently stronger version of) the inequality $\text{rank } G' \leq \text{rank } G - \rho_0(G)/2$ when G, G' are nilpotent of class 2 (or $c > 2$) and $r \in [F, F]$, which Steinberg [3] proves using other methods.

5. Theorems 1 and 2 and their corollaries can be proved in the category of pro- p -groups. (Here one allows only continuous homomorphisms and closed subgroups.) The arguments are completely analogous and even simpler and more elegant, by virtue of the Burnside basis theorem which holds for pro- p -groups [4, Chapter I]. The necessary cohomology is developed in [5]. In particular, the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ which appear in Theorem 1 are actual bases of F and F' . If $F(p)$ is the free pro- p -group on n generators, then the (finite) factor groups F/F_2 (in fact F/F_∞) and $F(p)/F(p)_2$ (respectively $F(p)/F(p)_\infty$) are isomorphic, as well as F/RF_2 and $F(p)/\bar{R}F(p)_2$, where $\bar{R}$ is the closed normal subgroup of $F(p)$ corresponding to R . (One can assume, if one wishes, that F is embedded in $F(p)$; then $\bar{R}$ is the (topological) closure of R in $F(p)$.) Now Remark 4 is valid in the setting of pro- p -groups; furthermore, the two "new versions" involving f_1 (of Remark 4) are equivalent by the preceding observation. It follows that Theorems 1 and 2 can be derived from their analogues in pro- p -groups (and vice versa).

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