ON GENERALIZING ALTERNATIVE RINGS

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ABSTRACT. Consider a ring R that satisfies the identity $(x, x, x) = 0$ and any two of the three identities: $(wx, y, z) + (w, x, [y, z]) - w(x, y, z) - (w, y, z)x = 0$; $([w, x], y, z) + (w, x, yz) - y(w, x, z) - (w, x, y)z = 0$; $(w, x \cdot y, z) - x \cdot (w, y, z) - y \cdot (w, x, z) = 0$. In this paper, we prove that if R has characteristic prime to 6 then R semiprime with idempotent e implies R has a Peirce decomposition in which the modules multiply as they do in an alternative ring. If in addition R is prime with idempotent $e \neq 0, 1$ then R is alternative.

1. Introduction. Let R be a nonassociative ring. For x, y, z in R define the associator $(x, y, z) = (xy)z - x(yz)$ and the commutator $[x, y] = xy - yx$. The ring R is power-associative if for each x in R , the subring generated by x is associative.

In this paper we will be concerned with rings that satisfy the identity

$$(1) \quad (x, x, x) = 0$$

and any two of the three identities:

$$(2) \quad g(w, x, y, z) = (wx, y, z) + (w, x, [y, z]) - w(x, y, z) - (w, y, z)x = 0;$$

$$(3) \quad h(w, x, y, z) = (w, x \cdot y, z) - x \cdot (w, y, z) - y \cdot (w, x, z) = 0;$$

$$(4) \quad k(w, x, y, z) = ([w, x], y, z) + (w, x, yz) - y(w, x, z) - (w, x, y)z = 0.$$

Rings satisfying (1), (2) and (4) were studied by Kleinfeld [2] and Smith [3]. Rings satisfying (1), (2) and (3) are anti-isomorphic to rings which satisfy (1), (3) and (4). This paper will concentrate on rings satisfying (1), (2) and (3). By anti-isomorphism our results will hold for rings satisfying (1), (3) and (4). These results are known for rings satisfying (1), (2) and (4) (see [3]).

From now on we assume R to be a ring satisfying (1), (2) and (3). It

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is known [1] that R is power-associative. For characteristic prime to 2 we define $x \cdot y = \frac{1}{2}(xy + yx)$. If R contains an idempotent then

$$R = R_1(e) + R_{\frac{1}{2}}(e) + R_0(e)$$

where $R_i(e) = \{x: x \cdot e = ix\}$. This is the Albert decomposition of R . We will denote $R_i(e)$ by R_i .

A nonassociative ring R is semiprime if R contains no nonzero ideal I such that $I^2 = 0$. The ring R is prime if R contains no nonzero ideals I and H such that $IH = 0$.

Theorem 1. *The following are equivalent for any nonassociative ring:*

- (a) R is prime.
- (b) R contains no nonzero ideals I and H such that $IH + HI = 0$.
- (c) R contains no nonzero ideals I and H such that $HI = 0$.

Proof. If R contains nonzero ideals I and H with $IH + HI = 0$ then $IH = 0$ so R is not prime. Thus (a) implies (b). To prove (b) implies (a) assume I, H are nonzero ideals with $IH = 0$ and assume R satisfies (b). Then $HI = IH + HI \neq 0$. Let $K = H \cap I$. We have $HI \subseteq K$ so $K \neq 0$. However, $K^2 \subseteq IH = 0$ so $K = 0$ a contradiction. We have proved (a) equivalent to (b). The proof that (b) and (c) are equivalent follows by symmetry.

Theorem 2. *Let R be semiprime. Then R is prime if and only if R contains no nonzero ideals I and H with $I \cap H = 0$.*

Proof. If $I \cap H = 0$ then $IH \subseteq I \cap H = 0$. If $IH = 0$, define $K = I \cap H$. We have $K^2 \subseteq IH = 0$ so $K = 0$.

2. Preliminaries. We assume throughout that R satisfies (1), (2) and (3) with characteristic prime to 6. The following identity holds in any ring.

$$(5) \quad t(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0.$$

If we subtract $g(w, x, y, z) - t(w, x, y, z) = 0$ we obtain

$$(6) \quad s(w, x, y, z) = (w, xy, z) - (w, x, zy) + (w, x, y)z - (w, y, z)x = 0.$$

We also note

$$(7) \quad g(w, x, y, y) = (wx, y, y) - w(x, y, y) - (w, y, y)x = 0.$$

From [1] we know that if e is an idempotent then $(e, R, e) = 0$. In the Albert decomposition $R_i R_{\frac{1}{2}} + R_{\frac{1}{2}} R_i \subseteq R_{\frac{1}{2}}$ for $i = 0, 1$. Defining B by $B =$

$\{x \text{ in } R_{1/2}: xR + Rx \subseteq R_{1/2}\}$ it is known [1] that B is an ideal in which each element squares to zero.

If R has a Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ ($R_{ij} = \{x: ex = ix, xe = jx\}$), then $R_{ij}R_{km} \subseteq \delta_{jk}R_{im}$ (Kronecker delta) with the following exceptions: $R_{ij}R_{ij} \subseteq R_{ji}$ ($i \neq j$); $R_{ii}R_{ji} \subseteq R_{ij}$. This latter exception is the most pathological. However, if we define

$$J = (R_{11}R_{01})R_{01} + R_{11}R_{01} + R_{00}R_{10} + (R_{00}R_{10})R_{10}$$

then J is an ideal and $J^2 = 0$ (see [1]).

3. Semiprime rings.

Lemma 3. *Under our hypotheses, the following hold:*

- (a) $(R_{1/2}^2, e, e) = (e, e, R_{1/2}^2) = 0$.
- (b) $([R, R], e, e) = (e, e, [R, R]) = 0$.
- (c) $(e, R, R), (R, e, R), (R, e, e)$ are in $R_{1/2}$.
- (d) (e, e, R) and (R, e, e) are in B .

Proof. We first note the following linearizations of (1):

- (8) $(x, y, z) + (y, z, x) + (z, x, y) + (z, y, x) + (y, x, z) + (x, z, y) = 0$.
- (9) $(x, y, y) + (y, y, x) + (y, x, y) = 0$.

Now (c) and (d) were proved in [1]. From (7) we have

$$g(e, e, y, z) = (e, y, z) + (e, e, [y, z]) - e(e, y, z) - (e, y, z)e = 0.$$

By (c), $(e, y, z) = e(e, y, z) + (e, y, z)e$ so $(e, e, [y, z]) = 0$. Since $(e, R, e) = 0$, $(e, [R, R], e) = 0$ so (9) implies $([y, z], e, e) = 0$ and (b) is established. Let x and y be in $R_{1/2}$. Clearly $xy = x \cdot y + \frac{1}{2}[x, y]$. It is known [1] that $x \cdot y$ is in $R_1 + R_0$ so $(x \cdot y, e, e) = (e, e, x \cdot y) = 0$. Thus

$$(xy, e, e) = (x \cdot y, e, e) + \frac{1}{2}([x, y], e, e) = 0.$$

By symmetry, $(e, e, xy) = 0$ and (a) is proved.

Theorem 4. *Let $I = (R, e, e)$. Then I is an ideal with $[R, I] = I^2 = R_{1/2}I = 0$.*

Proof. Let $I' = (e, e, R)$. From (9) and the fact that $(e, R, e) = 0$ we have $I = I'$. Now, let $b = (x, e, e)$. From Lemma 3(d), $b \in B$ so $be + eb = b$. We note, from (5) and the fact that $(e, R, e) = 0$,

$$0 = t(e, e, x, e)$$

$$(10) \quad \begin{aligned} &= (e, x, e) - (e, ex, e) + (e, e, xe) - e(e, x, e) - (e, e, x)e \\ &= (e, e, xe) - (e, e, x)e. \end{aligned}$$

Substituting $w = x = y = e$ and $z = x$ in (6) yields

$$(11) \quad \begin{aligned} 0 &= s(e, e, e, x) \\ &= (e, e, x) - (e, e, xe) + (e, e, e)x - (e, e, x)e \\ &= (e, e, x) - (e, e, xe) - (e, e, x)e. \end{aligned}$$

If we add (10) and (11) we have $(e, e, x) = 2(e, e, x)e$. Since b is in I' for $I = I'$ this means $b = 2be$ so $be = eb = \frac{1}{2}b$. We now choose b and b' from $I \subseteq B \subseteq R_{\frac{1}{2}}$. Lemma 3(a) implies $(bb', e, e) = 0$ so

$$(12) \quad \begin{aligned} 0 &= g(b, b', e, e) \\ &= (bb', e, e) + (b, b', [e, e]) - b(b', e, e) - (b, e, e)b' \\ &= -b(b', e, e) - (b, e, e)b'. \end{aligned}$$

Since, for b in I , $be = eb = \frac{1}{2}b$, $(b, e, e) = \frac{1}{2}be - \frac{1}{2}b = \frac{1}{4}b - \frac{1}{2}b = -\frac{1}{4}b$.

Therefore, from (12), $0 = -b(b', e, e) - (b, e, e)b' = \frac{1}{4}bb' + \frac{1}{4}bb' = \frac{1}{2}bb'$. We have proved $I^2 = 0$.

Define $K = (I, e, e)$. As above we easily see $K = (e, e, I)$. Let $b = (x, e, e)$. Then $(b, e, e) = -\frac{1}{4}b$ as above so b is in K . Therefore $K = I$ and we have proved $I = (I, e, e) = (e, e, I)$. From (7) we have the important fact

$$(13) \quad (wx, e, e) = w(x, e, e) + (w, e, e)x.$$

If w is in $R_1 + R_0$ then $(w, e, e) = 0$ and $w(x, e, e)$ is in I . Therefore $(R_1 + R_0)I \subseteq I$. Similarly $I(R_1 + R_0) \subseteq I$. Now let w be in $R_{\frac{1}{2}}$. By Lemma 3(a), $(w(x, e, e), e, e) = 0$ since (x, e, e) is in $I \subseteq R_{\frac{1}{2}}$. Since $I^2 = 0$, $(w, e, e)(x, e, e) = 0$ so from (13) we have $w((x, e, e), e, e) = 0$. But, if $b = (x, e, e)$ is in I then $w(b, e, e) = -\frac{1}{4}wb$ so $wb = 0$ and $R_{\frac{1}{2}}I = 0$. Similarly $IR_{\frac{1}{2}} = 0$. Thus, I is an ideal of R and $R_{\frac{1}{2}}I = 0$.

To see that $[R, I] = 0$ we first note that $[R_{\frac{1}{2}}, I] = 0$ was proved above. If w is in $R_1 + R_0$ then $(w, e, e) = 0$. By (13), $(wx, e, e) = w(x, e, e)$ and $(xw, e, e) = (x, e, e)w$. From Lemma 3(b) we have $0 = ([w, x], e, e) = [w, (x, e, e)]$ so $[R, I] = 0$.

Theorem 5. *Let R satisfy (1), (2) and (3) with characteristic prime to 6. If e is an idempotent in R and if R is semiprime then R has a Peirce decomposition relative to e . Furthermore the modules R_{ij} multiply as in an alternative ring ($R_{ij}R_{km} \subseteq \delta_{jk}R_{im}$ except $R_{ij}^2 \subseteq R_{ji}$ for $i \neq j$) and $T = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$ is an alternative ideal.*

Proof. From Theorem 4 and its proof, $I = (R, e, e) = (e, e, R)$ is such that $I^2 = 0$ and I is an ideal. Therefore $I = 0$. Since $(e, R, e) = 0$ then R has a Peirce decomposition. Also by semiprimeness, the ideal J defined in §2 is zero so $R_{ij}R_{km} \subseteq \delta_{jk}R_{im}$ except that for $i \neq j$ $R_{ix}^2 \subseteq R_{ji}$. The arguments in [1] now show that T is an alternative ideal.

4. Prime rings.

Lemma 6. *If R is semiprime then $(R_{ji}, R_{ii}, R_{ii}) = (R_{ii}, R_{ii}, R_{ij}) = 0$ for $i \neq j$.*

Proof. We will use letters with subscripts to denote elements in R_{ij} , etc. For example, w_{10} in R_{10} . We compute

$$\begin{aligned} 0 &= h(w_{01}, e, y_{11}, z_{11}) \\ &= (w_{01}, e \cdot y_{11}, z_{11}) - e \cdot (w_{01}, y_{11}, z_{11}) - y_{11} \cdot (w_{01}, e, z_{11}). \end{aligned}$$

However, $(w_{01}, e, z_{11}) = 0$, (w_{01}, y_{11}, z_{11}) is in R_{01} . Therefore, $0 = (w_{01}, y_{11}, z_{11}) - \frac{1}{2}(w_{01}, y_{11}, z_{11}) = \frac{1}{2}(w_{01}, y_{11}, z_{11})$. Again using (3), we obtain

$$\begin{aligned} 0 &= h(w_{10}, e, y_{00}, z_{00}) \\ &= (w_{10}, e \cdot y_{00}, z_{00}) - e \cdot (w_{10}, y_{00}, z_{00}) - y_{00} \cdot (w_{10}, e, z_{00}) \\ &= -\frac{1}{2}(w_{10}, y_{00}, z_{00}). \end{aligned}$$

Therefore $(R_{ji}, R_{ii}, R_{ii}) = 0$ for $i \neq j$. Using $0 = h(z_{11}, y_{11}, e, w_{10}) = h(z_{00}, y_{00}, e, w_{01})$ we can similarly prove $(R_{ii}, R_{ii}, R_{ij}) = 0$ for $i \neq j$.

Definition. Define $P_i = (R_{ii}, R_{ii}, R_{ii})$ and $S_i = P_i + P_i R_{ii}$ for $i = 0, 1$.

Lemma 7. *If R is semiprime then $R_{ij}S_i = S_i R_{ji} = R_{jj}S_i = S_i R_{jj} = R_{ji}S_i = S_i R_{ij} = 0$ for $i \neq j$.*

Proof. Since $S_i \subseteq R_{ii}$, $R_{ij}S_i = S_i R_{ji} = R_{jj}S_i = S_i R_{jj} = 0$ follows from the multiplication of the modules R_{ij} . Compute

$$\begin{aligned} 0 &= g(w_{11}, x_{10}, y_{11}, z_{11}) \\ &= (w_{11}x_{10}, y_{11}, z_{11}) + (w_{11}, x_{10}, [y_{11}, z_{11}]) - w_{11}(x_{10}, y_{11}, z_{11}) - (w_{11}, y_{11}, z_{11})x_{10} \\ &= - (w_{11}, y_{11}, z_{11})x_{10} \end{aligned}$$

since $R_{10}R_{11} = 0$. Therefore $P_1R_{10} = 0$. From $0 = g(w_{00}, x_{01}, y_{00}, z_{00})$ we show similarly that $P_0R_{01} = 0$. Applying Lemma 6 we have for $i \neq j$, $S_i R_{ij} = P_i R_{ij} + (P_i R_{ii})R_{ij} = P_i(R_{ii}R_{ij}) \subseteq P_i R_{ij} = 0$.

To prove $R_{01}P_1 = 0$, write

$$\begin{aligned} 0 &= b(w_{11}, x_{01}, y_{11}, z_{11}) \\ &= (w_{11}, x_{01} \cdot y_{11}, z_{11}) - x_{01} \cdot (w_{11}, y_{11}, z_{11}) - y_{11} \cdot (w_{11}, x_{01}, z_{11}) \\ &= -\frac{1}{2}x_{01}(w_{11}, y_{11}, z_{11}) \end{aligned}$$

since $R_{11}R_{01} = 0$. Also, $0 = b(w_{00}, x_{10}, y_{00}, z_{00})$ implies $R_{10}P_0 = 0$. By Lemma 6 we have for $i \neq j$, $R_{ij}S_j = R_{ij}P_j + R_{ij}(P_jR_{jj}) = (R_{ij}P_j)R_{jj} = 0$.

Theorem 8. *If R is a semiprime ring satisfying (1), (2), and (3) with characteristic prime to 6 then S_1 and S_0 are ideals. In addition, $S_1S_0 = TS_1 = (TS_0)R_{00} = 0$.*

Proof. We first prove that each S_i is an ideal. From Lemma 7, all that is needed is to show that $S_iR_{ii} + R_{ii}S_i \subseteq S_i$. Clearly, $S_iR_{ii} = P_iR_{ii} + (P_iR_{ii})R_{ii} \subseteq P_iR_{ii}$ (by Lemma 6) $\subseteq S_i$. Using (2), we have

$$\begin{aligned} 0 &= g(w_{ii}, x_{ii}, y_{ii}, z_{ii}) \\ (14) \quad &= (w_{ii}x_{ii}, y_{ii}, z_{ii}) + (w_{ii}, x_{ii}, [y_{ii}, z_{ii}]) - w_{ii}(x_{ii}, y_{ii}, z_{ii}) - (w_{ii}, y_{ii}, z_{ii})x_{ii}. \end{aligned}$$

Define $D_i = P_i + R_{ii}P_i$ for $i = 0, 1$. By (14), $D_i \subseteq S_i$ and $S_i \subseteq D_i$ so $D_i = S_i$. Therefore $R_{ii}S_i = R_{ii}D_i = R_{ii}P_i + R_{ii}(R_{ii}P_i) \subseteq R_{ii}P_i$ (by Lemma 6) $\subseteq D_i = S_i$.

To see that $S_1S_0 = 0$ we need only note that $S_1S_0 \subseteq R_{11}R_{00} = 0$.

Before proceeding we must prove that $(R_{ij}, R_{ji}, R_{ii})R_{ii} = 0$ for $i \neq j$. We have

$$\begin{aligned} 0 &= g(w_{ij}, x_{ii}, y_{ji}, z_{ii}) \\ &= (w_{ij}x_{ii}, y_{ji}, z_{ii}) + (w_{ij}, x_{ii}, [y_{ji}, z_{ii}]) - w_{ij}(x_{ii}, y_{ji}, z_{ii}) - (w_{ij}, y_{ji}, z_{ii})x_{ii} \\ &= - (w_{ij}, y_{ji}, z_{ii})x_{ii} \end{aligned}$$

since $R_{ii}R_{ji} = R_{ij}R_{ii} = 0$. Indeed, $(w_{10}, y_{01}, z_{11}) = (w_{10}, y_{01}, z_{11})e = 0$. Consequently, $(R_{10}R_{01})S_1 = R_{10}(R_{01}S_1) = 0$ by Lemma 7. Using the definition of T and Lemma 7 we conclude that $TS_1 = 0$. Finally, from $(w_{01}, y_{10}, z_{00})x_{00} = 0$, the definition of T and Lemma 7, $(TS_0)R_{00} = 0$.

Theorem 9. *If R is a prime ring satisfying (1), (2) and (3) with characteristic prime to 6 and if R contains an idempotent $e \neq 1$ then R is alternative.*

Proof. If $T = 0$ then $R = R_{11} \oplus R_{00}$ so $R_{00} = 0$ and e is the unity

element contrary to assumption. Hence, $T \neq 0$. From Theorem 8, $TS_1 = 0$ so $S_1 = 0$. From the definition of T and Lemma 7, $TS_0 \subseteq (R_{01}R_{10})S_0$. Define $T_0 = R_{01}R_{10}$. Now, S_0 is an ideal so $TS_0 \subseteq S_0$. Therefore, by Lemma 7, $(TS_0)R = (TS_0)R_{00}$ and $R(TS_0) = R_{00}(TS_0)$. Using Theorem 8, $(TS_0)R_{00} = 0$ so $(TS_0)R = 0$. To show TS_0 is an ideal we need only have $R(TS_0) \subseteq TS_0$. But, $R(TS_0) = R_{00}(T_0S_0) = R_{00}((R_{01}R_{10})S_0)$. From (3),

$$\begin{aligned} 0 &= h(w_{00}, x_{01}, y_{10}, s_0) \\ &= (w_{00}, x_{01} \cdot y_{10}, s_0) - x_{01} \cdot (w_{00}, y_{10}, s_0) - y_{10} \cdot (w_{00}, x_{01}, s_0) \\ &= \frac{1}{2}(w_{00}, x_{01}y_{10}, s_0). \end{aligned}$$

Therefore $R_{00}(T_0S_0) = (R_{00}T_0)S_0 \subseteq T_0S_0$ since T is an ideal. Therefore, TS_0 is an ideal. Another application of Theorem 8 gives $(TS_0)S_0 = 0$ so $TS_0 = 0$ or $S_0 = 0$. But, if $TS_0 = 0$ then $S_0 = 0$ since $T \neq 0$.

We conclude that $S_1 = S_0 = 0$. Therefore R_{11} and R_{00} are associative and the proof in [1] that T is alternative can be repeated to show that R is alternative. All they needed was that R_{11} and R_{00} be associative.

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