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LEBESGUE MEASURE IS A REPRESENTING MEASURE[1]

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ABSTRACT. Lebesgue measure on the unit interval I is multiplicative on some maximal Dirichlet algebra on I. Related results are obtained.

The main point of the present note is the observation that Lebesgue measure on the unit interval $I = [0, 1]$ is multiplicative on some uniform algebra on I, which answers a question which has apparently circulated for some time, and was posed to me by my colleague G. M. Leibowitz.

Theorem. *If μ is a nonatomic (Borel) probability measure on I whose closed support is all of I, then μ is multiplicative on some maximal (proper) Dirichlet subalgebra of $C(I)$.*

Proof. If J is an arc in the complex plane $\mathbf{C}$, we consider the algebra A, first studied by J. Wermer [4], of functions continuous on the Riemann sphere $S^2 = \mathbf{C} \cup \{\infty\}$ and holomorphic on $U = S^2 \backslash J$. It has been shown by A. Browder and J. Wermer [1] that J can be so chosen that A is a uniform algebra whose Šilov boundary is J, and $A|J$ is a maximal (proper) Dirichlet algebra on J. Pick z in U and let ν denote the representing measure for z on J. Then ν is nonatomic, since any atom of ν would lie in the Gleason part for A which contains z, whereas all points of J are peak points for A. Further, the closed support of ν is all of J. For let $x \in J$ and let V be an open set in S^2 containing x. There is f in A such that $f(x) = \sup |f| = 1$ while $|f| < 1/3$ on $S^2 \backslash V$. Take $z' \in U$ so close to x that $|f(z')| > 2/3$. If ν' denotes the representing measure for z' on J, then ν and ν' are (boundedly) equivalent measures, because z and z' lie in the same Gleason part for A. But clearly $\int f \, d\nu' = f(z')$ implies that ν', hence ν, has some mass on $J \cap V$.

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Let τ denote a homeomorphism of I onto J. Define functions g, h: $I \to I$ by $g(t) = \nu(\tau([0, t]))$ and $h(t) = \mu([0, t])$. These are homeomorphisms of I onto itself, and μ is multiplicative on the maximal Dirichlet algebra $\{f \circ \tau \circ g^{-1} \circ h : f \in A\}$ on I. Q.E.D.

Thus Lebesgue measure is even a Jensen measure.

The heart of the above argument is existence of a nonatomic multiplicative measure ν whose support is precisely J. As the following theorem shows, this existence can be recovered if we know simply that J is the Šilov boundary for A, which happens, e.g., if J has locally positive measure (cf. [3, 7.9]); of course, this entails replacing "maximal Dirichlet" by "uniform" in the statement of the preceding theorem, though the measure will remain an Arens-Singer measure because A is known to satisfy $A^{-1} = \exp(A)$.

Theorem. *Let A be a uniform algebra on the compact metric space X and let π be a Gleason part for A which is not contained in X. Then any $z \in \pi$ has a nonatomic representing measure on X whose closed support contains every peak point for A which lies in the closure of $\pi \backslash X$.*

Proof. Let $\{z_n\}$ denote a dense sequence in $\pi \backslash X$ (repetitions allowed in case $\pi \backslash X$ is finite). There are (strictly) positive constants b_n such that $u(z) - b_n u(z_n) \geq 0$ whenever $u \in \mathrm{Re}(A)$ is nonnegative, so by Choquet's theorem (cf. [2]) there is a positive (Borel) measure σ_n on X supported by P, the set of peak points for A, such that

$$\int f \, d\sigma_n = f(z) - b_n f(z_n) \tag{1}$$

for every $f \in A$. Let ν_n be a Jensen measure for z_n on X, i.e., a representing measure such that $\log|f(z_n)| \leq \int \log|f| \, d\nu_n$ for all $f \in A$. It is immediate that ν_n is nonatomic.

The measure $\nu = \Sigma_1^\infty 2^{-n}(\sigma_n + b_n \nu_n)$ is, by (1), a representing measure for z. If it had an atom x, then x and z would lie in the same Gleason part for A; on the other hand, since the ν_n are nonatomic, x would be an atom for some σ_n, hence $x \in P$, a contradiction. Thus ν is nonatomic. Finally, let $x \in P$ lie in the closure of $\pi \backslash X$. If V is a neighborhood of x in the spectrum of A, we can argue as in the proof of the preceding theorem to see that for z_n close to x, ν_n (and so ν) has some mass on $X \cap V$. Thus ν has all the required properties. Q.E.D.

If X is not metrizable, the theorem will still hold provided $\pi \backslash X$ is separable, or at least contains a sequence whose closure contains all (generalized) peak points lying in the closure of $\pi \setminus X$. In this case σ_n is selected to be a "maximal" measure (cf. [2]) and some extra care is required in working out the details.

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