A LARGE ALGEBRAICALLY CLOSED FIELD

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ABSTRACT. A sequence is called an R.R.S. sequence if, roughly speaking, it is generated by some member of a set of recurrence formulas over the field $Q(i)$ which involves only rational operations. It is proved that the set of limits of all convergent R.R.S. sequences forms a countable algebraically closed field. Moreover, the field is shown to contain all numbers of the form e^α , where α is an algebraic number.

Let $Q(i)$ denote the field of rationals with i adjoined. We define an m th order rational recursion scheme (R.R.S) to be one of the form $z_1(0), \dots, z_m(0) \in Q(i)$ (initial conditions) and

$$\begin{aligned} z_1(n+1) &= R_1(z_1(n), \dots, z_m(n)) \\ &\vdots \\ z_m(n+1) &= R_m(z_1(n), \dots, z_m(n)) \quad (\text{recursion formulae}) \end{aligned}$$

where $R_1, \dots, R_m$ are rational functions of at most m variables with coefficients from $Q(i)$. An R.R.S. sequence is one of the form $\{z_j(n)\}$ where $n = 0, 1, 2, \dots$ for some fixed j with $1 \leq j \leq m$.

The set of all limits of convergent R.R.S. sequences forms a field which is denoted by $Q^*(i)$. $Q^*(i)$ is a countably infinite field, since the number of R.R.S. sequences is countable. We will prove that $Q^*(i)$ is algebraically closed, and that $Q^*(i)$ is large in that it contains all numbers of the form e^α where α is an algebraic number. We note that the existence of fields like $Q^*(i)$ could be established by taking the algebraic closure of the rationals with all numbers of the form e^α adjoined.

Proposition 1. $Q^*(i)$ is algebraically closed.

Proof. Let $P(z) = a_r z^r + a_{r-1} z^{r-1} + \dots + a_1 z + a_0$ be a polynomial

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with coefficients in $Q^*(i)$ having α as a root. Assume, without loss of generality, that α is a simple root. It suffices to show that α is in $Q^*(i)$.

Let $\{a_j(n)\}$, $n = 0, 1, 2 \dots$, be an R. R. S. sequence converging to a_j where $0 \leq j \leq r$. Let $P_n(z) = a_r(n)z^r + a_{r-1}(n)z^{r-1} + \dots + a_0(n)$. Note that $|a_j(0) - a_j|$ can be made arbitrarily small by adjusting the initial conditions of the R.R.S.

We will show that $z(n+1) = z(n) - (P_n(z(n))/P'_n(z(n)))$ is an R.R.S. sequence converging to α , subject to choosing $a_j(0)$ for $0 \leq j \leq r$ and $z(0)$ appropriately. (Here $P'_n(z(n))$ is the derivative of $P_n(z(n))$ with respect to $z(n)$.)

Let D be a disc with center α and radius less than or equal to $1/2^k$ with k an integer greater than zero, so that $|P'_n(z)| > 2m$ for $z \in D$ where m is a positive real number.

Let

$$\begin{aligned} g_n(z) = & a_r(n)(z^{r-2} + z^{r-3}(z + \alpha) + \dots + (z^{r-2} + z^{r-3}\alpha + \dots + \alpha^{r-2})) \\ & + a_{r-1}(n)(z^{r-3} + z^{r-4}(z + \alpha) + \dots + (z^{r-3} + z^{r-4}\alpha + \dots + \alpha^{r-3})) \\ & \cdot \\ & \cdot \\ & + a_3(n)(z + (z + \alpha)) + a_2(n). \end{aligned}$$

Now choose $a_j(0)$ for $0 \leq j \leq r$ so that $|P'_n(z)| \geq m$ for $z \in D$ and $|P_n(\alpha)|/|P'_n(z)| \leq 2^{-(k+s+4)}$ for $z \in D$, where s is some positive integer so that $|g_n(z)|/|P'_n(z)| \leq 2^s$ for $z \in D$, and choose $z(0)$ in $Q(i)$ so that $|z(0) - \alpha| \leq 2^{-(k+s+2)}$.

It is easy to show that

$$z(n+1) - \alpha = (z(n) - \alpha)^2 \cdot (g_n(z(n))/P'_n(z(n))) - P_n(\alpha)/P'_n(z(n))$$

for $n = 0, 1, 2 \dots$. So we have

$$|z(n+1) - \alpha| \leq |z(n) - \alpha|^2 |g_n(z(n))/P'_n(z(n))| + |P_n(\alpha)/P'_n(z(n))|$$

for $n = 0, 1, 2 \dots$. By induction, we have $|z(n) - \alpha| \leq 2^{-(k+s+3)}$ for $n \geq 1$. Since the sequence $\{P_n(\alpha)\}$ converges to zero, we can find an integer N_1 so that for $n \geq N_1$, $|P_n(\alpha)/P'_n(z(n))| \leq 2^{-(k+s+6)}$. It follows that $|z(n) - \alpha| \leq 2^{-(k+s+5)}$ for $n \geq N_1$. This process can be repeated as often as one wants, so $\{z(n)\}$ converges to α . Thus α is in $Q^*(i)$, and the proof of Proposition 1 is complete.

Now suppose α is an algebraic number of degree n . Since $Q^*(i)$ is

algebraically closed we have that α is in $Q^*(i)$. We define rational numbers $b_j(k)$ for $0 \leq j \leq n-1$ to be the numbers which satisfy

$$\alpha^k = b_0(k) + b_1(k)\alpha + \dots + b_{n-1}(k)\alpha^{n-1} \quad \text{for } k = 0, 1, 2, \dots$$

It is clear that $b_0(0) = 1$ and that

$$(1) \quad \alpha^{k+1} = b_0(k)\alpha + b_1(k)\alpha^2 + \dots + b_{n-2}(k)\alpha^{n-1} + b_{n-1}(k)\alpha^n.$$

By substituting $\alpha^n = b_0(n) + b_1(n)\alpha + \dots + b_{n-1}(n)\alpha^{n-1}$ in (1) we get

$$\begin{aligned} \alpha^{k+1} &= b_n(k) \cdot b_0(n) + (b_0(k) + b_{n-1}(k)b_1(n))\alpha + \dots \\ &\quad + (b_{n-2}(k) + b_{n-1}(k)b_{n-1}(n))\alpha^{n-1}. \end{aligned}$$

So if we assume that $b_{-1}(k) = 0$ we get

$$(2) \quad b_j(k+1) = b_{j-1}(k) + b_{n-1}(k)b_j(n)$$

for $0 \leq j \leq n-1$ and $k = 0, 1, 2, \dots$

From (2) we conclude that $b_j(1) \leq 2M^2$ for $0 \leq j \leq n-1$ where $M = \max\{1, b_0(n), \dots, b_{n-1}(n)\}$ and by an induction

$$(3) \quad b_j(k) \leq 2^k M^{k+1} \quad \text{for } 0 \leq j \leq n-1.$$

Proposition 2. *If α is an algebraic number of degree n then e^α is in $Q^*(i)$.*

Proof. It is known that $e^\alpha = \sum_{k=0}^{\infty} \alpha^k/k!$. From (3) we get that $\sum_{k=0}^{\infty} b_j(k)/k!$ is absolutely convergent so it is true that $e^\alpha = \sum_{j=0}^{n-1} \alpha^j \sum_{k=0}^{\infty} b_j(k)/k!$. Since α is in $Q^*(i)$, if we can show that $\sum_{k=0}^{\infty} b_j(k)/k!$ is in $Q^*(i)$ for $0 \leq j \leq n-1$ we will have e^α in $Q^*(i)$. It suffices to show that there is an R.R.S. sequence converging to $\sum_{k=0}^{\infty} b_j(k)/k!$ for $0 \leq j \leq n-1$.

We exhibit an R.R.S. of order $3n+4$ with initial conditions

$$\begin{aligned} z_j(0) &= b_j(0) \quad \text{for } 0 \leq j \leq n-1, \\ z_j(0) &= b_{j-n}(n) \quad \text{for } n \leq j \leq 2n-1, \\ z_{2n}(0) &= 0, \\ z_j(0) &= b_{j-(2n+1)}(1) \quad \text{for } 2n+1 \leq j \leq 3n, \\ z_{3n+1}(0) &= 1, \quad z_{3n+2}(0) = 1, \quad z_{3n+3}(0) = 2, \end{aligned}$$

and recursion formulae

$$\begin{aligned}
z_j(k) &= z_j(k-1) + z_{j+2n+1}(k-1) \cdot z_{3n+2}(k-1) \quad \text{for } 0 \leq j \leq n-1, \\
z_j(k) &= z_j(k-1) \quad \text{for } n \leq j \leq 2n-1, \\
z_{2n}(k) &= z_{2n}(k-1), \\
z_k(k) &= z_{j-1}(k-1) + z_{j-(n+1)}(k-1) \cdot z_{3n}(k-1) \quad \text{for } 2n+1 \leq j \leq 3n, \\
z_{3n+1}(k) &= z_{3n+1}(k-1), \\
z_{3n+2}(k) &= z_{3n+2}(k-1)/z_{3n+3}(k-1), \\
z_{3n+3}(k) &= z_{3n+3}(k-1) + z_{3n+1}(k-1).
\end{aligned}$$

It can be shown by induction that

$$\begin{aligned}
z_j(k) &= \sum_{m=0}^k \frac{b_j(m)}{m!} \quad \text{for } 0 \leq j \leq n-1, \\
z_j(k) &= b_{j-n}(n) \quad \text{for } n \leq j \leq 2n-1, \\
z_{2n}(k) &= b_{-1}(k+1) = 0, \\
z_j(k) &= b_{j-(2n+2)}(k) + b_{j-(2n+1)}(n) \cdot b_{n-1}(k) \\
&= b_{j-(2n+1)}(k+1) \quad \text{for } 2n+1 \leq j \leq 3n, \\
z_{3n+1}(k) &= 1, \quad z_{3n+2}(k) = \frac{1}{(k+1)!}, \quad z_{3n+3}(k) = k+2.
\end{aligned}$$

Thus the sequence $\{z_j(k)\}$ converges to $\sum_{k=1}^{\infty} b_j(k)/k!$ for $0 \leq j \leq n-1$, and the sum is therefore in $Q^*(i)$. This completes the proof of Proposition 2.

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