ϵ -CONTINUITY AND SHAPE

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ABSTRACT. In this paper a relationship is established between the concepts of ϵ -continuity and shape for compact metric spaces, thus answering a question raised by Victor Klee.

Victor Klee [4] has asked whether there is a relationship between ϵ -continuity and shape. Here we show that, in the category of compact metric spaces, the shape morphisms can be identified in a one-to-one manner with the limits of certain equivalence classes of ϵ -continuous functions.

Throughout this paper all spaces are assumed to be compact metric spaces. If f is a continuous function (map) from one space X to another space Y , $[f]$ will denote the homotopy equivalence class of f . By $\text{Cov}(Y)$ we will denote the set of all finite open covers of Y directed by refinement. If A and B are collections of subsets of Y , $A < B$ means A refines B . We will use I to denote the unit interval $[0, 1]$. A function f (not necessarily continuous) from a space X to a space Y is α -continuous, $\alpha \in \text{Cov}(Y)$, if there is a $\beta \in \text{Cov}(X)$ such that $\{\{f(x), f(x')\}: x, x' \in V \in \beta\} < \alpha$; (f is ϵ -continuous in the sense of Klee, for some $\epsilon > 0$, if and only if it is α -continuous for some $\alpha \in \text{Cov}(Y)$ consisting of open balls of radius ϵ). If g is another function from X to Y , f and g are α -homotopic if there is an α -continuous function H from $X \times I$ to Y such that, for each $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. This defines an equivalence relation on the set of α -continuous functions from X to Y . We say that f and g are α -close if $\{\{f(x), g(x)\}: x \in X\} < \alpha$.

For $\alpha \in \text{Cov}(Y)$, $[X, Y]_\alpha$ denotes the set of α -homotopy classes of α -continuous functions f from X to Y . The equivalence class of such an f is denoted $[f]_\alpha$. By $\lim [X, Y]_\alpha$ we denote the usual representation, in the category of sets and functions, of the limit of the inverse system $\{[X, Y]_\alpha; p_{\alpha\alpha'}; \alpha' < \alpha \in \text{Cov}(Y)\}$ where $p_{\alpha\alpha'}([f]_{\alpha'}) = [f]_\alpha$ for $\alpha' < \alpha \in \text{Cov}(Y)$.

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We will establish a bijection from $\lim[X, Y]_\alpha$ onto the set of all shape morphisms from X to Y . In the course of the proof we obtain Lemma 3: If P is an ANR there is an $\alpha \in \text{Cov}(P)$ such that any two continuous functions from a space X to P which are α -homotopic are homotopic.

For $\alpha \in \text{Cov}(Y)$, $\nu(\alpha)$ will denote the geometric realization of the nerve of α . For $U \in \alpha$, U will also denote a vertex of $\nu(\alpha)$ and $\text{st}(U)$ will denote the star of the vertex U in $\nu(\alpha)$. By $\text{st}(\alpha)$ we denote the open cover of $\nu(\alpha)$, $\{\text{st}(U) : U \in \alpha\}$. By ϕ_α we denote a canonical map from Y to $\nu(\alpha)$. This means that there is a partition of unity $\{\phi_U : U \in \alpha\}$ such that U contains the cozero set of ϕ_U for all $U \in \alpha$, and $\phi_\alpha(y) = \sum \{\phi_U(y)U : U \in \alpha\}$ for $y \in Y$. For $\alpha' < \alpha \in \text{Cov}(Y)$, $\phi_{\alpha\alpha'}$ will denote a canonical map from $\nu(\alpha')$ to $\nu(\alpha)$. That is, $\phi_{\alpha\alpha'}$ is a simplicial map determined by assigning to each $U' \in \alpha'$ a $\phi_{\alpha\alpha'}(U') \in \alpha$ containing U' . Whatever canonical maps are chosen we always have $[\phi_\alpha] = [\phi_{\alpha\alpha'}\phi_{\alpha'}]$ for $\alpha' < \alpha \in \text{Cov}(Y)$.

A shape morphism F from X to Y will be interpreted in the sense of Holsztyński [2]. That is, for each ANR P and each homotopy class of maps $[g]$ from Y to P , $F([g])$ is a homotopy class of maps from X to P . In addition, if Q is an ANR, h a map from Y to Q , and p a map from P to Q such that $[h] = [pg]$, then $F([h]) = [p]F([g])$. Let S denote the shape functor. It can be shown (see [1, p. 356, 2.12, and p. 360, Exercise 5]) that, in the shape category, $\{Y; S(\phi_\alpha); \alpha \in \text{Cov}(Y)\}$ is a presentation of Y as the limit of the inverse system $\{\nu(\alpha); S(\phi_{\alpha\alpha'}); \alpha' < \alpha \in \text{Cov}(Y)\}$. It follows that a set $\{F([\phi_\alpha]) : \alpha \in \text{Cov}(Y)\}$, with $F([\phi_\alpha])$ a homotopy class of maps from X to $\nu(\alpha)$ and $F([\phi_\alpha]) = [\phi_{\alpha\alpha'}]F([\phi_{\alpha'}])$ for all $\alpha' < \alpha \in \text{Cov}(Y)$, determines a unique shape morphism F from X to Y .

1. Definition. Suppose $f = \lim \{[f_\alpha]_\alpha : \alpha \in \text{Cov}(Y)\} \in \lim[X, Y]_\alpha$ and F is a shape morphism from X to Y . Then f and F are associated if, for each $\alpha \in \text{Cov}(Y)$ and each $g_\alpha \in F([\phi_\alpha])$, g_α and $\phi_\alpha f_\alpha$ are $\text{st}(\alpha)$ -homotopic.

In §§ 4, 5, and 6 of this paper, this association is shown to be a bijection. First we give two lemmas.

2. Lemma. Suppose P is an ANR and $\beta \in \text{Cov}(P)$. There is a $\gamma \in \text{Cov}(P)$ such that, given f γ -continuous from a space X to P , there is an $\alpha \in \text{Cov}(X)$ and a continuous g from $\nu(\alpha)$ to P satisfying

- (i) $\text{st}(\alpha) < g^{-1}(\beta)$,
- (ii) f and $g\phi_\alpha$ are β -homotopic,
- (iii) f and $g\phi_\alpha$ are β -close.

Proof. Consider P as a closed subspace of the Hilbert cube I^∞ .

There is a neighborhood W of P in I^∞ and a retraction r of W onto P . Let γ' be a cover of W by convex open sets, which refines $r^{-1}(\beta)$. Then let γ'' be an open cover of W which star-refines γ' . Finally, let $\gamma \in \text{Cov}(P)$ which star-refines γ'' restricted to P . Suppose f is a γ -continuous function from X to P . There is an $\alpha = \{U_1, \dots, U_n\} \in \text{Cov}(X)$ such that for each i , $1 \leq i \leq n$, there is a $V_i'' \in \gamma''$ containing $f(U_i)$. Let $V_i' \in \gamma'$ contain the star of V_i'' with respect to γ'' . Fix $x_i \in U_i$ for $1 \leq i \leq n$. Suppose $y = \sum\{t_i U_i: 1 \leq i \leq n\} \in \nu(\alpha)$. Define $g'(y) = \sum\{t_i f(x_i): 1 \leq i \leq n\}$. Then $g'(y) \in W$ since $\bigcap\{U_i: t_i \neq 0\}$ nonempty implies $\{f(x_i): t_i \neq 0\}$ is contained in V_j' for any j with $t_j \neq 0$, and V_j' is convex. Let $g = rg'$. Since $g'(\text{st}(U_i))$ is contained in V_i' for $1 \leq i \leq n$, (i) holds. Furthermore, if $x \in U_i$, then $f(x)$ and $g'(\phi_\alpha(x)) \in V_i'$ so f and $g\phi_\alpha$ are β -close and β -homotopic by the β -homotopy H from $X \times I$ to P defined as: $H(x, t) = r(tg'\phi_\alpha(x) + (1-t)f(x))$ for $x \in X$, $t \in I$.

3. Lemma. *If P is an ANR there is an $\alpha \in \text{Cov}(P)$ such that any two continuous functions from a space X to P which are α -homotopic are homotopic.*

Proof. First note that the statement holds if " α -homotopic" is replaced by " α -close". (See [3, Theorem 1.1, p. 111].) Suppose α' is such a cover. Let $\alpha'' \in \text{Cov}(P)$ doubly star-refine α' and choose $\alpha \in \text{Cov}(P)$ to refine α'' and to satisfy the previous lemma with α'' in place of β and α in place of γ . Suppose g and h are continuous functions from a space X to P which are α -homotopic. Then there is a finite sequence of α -continuous functions $\{f_i: 0 \leq i \leq n\}$ from X to P such that $f_0 = g$, $f_n = h$, and f_i is α -close to f_{i-1} for $1 \leq i \leq n$. Using §2, choose g_i continuous with g_i α'' -close to f_i for $1 \leq i \leq n-1$. Let $g_0 = g$ and $g_n = h$. Then g_i is α' -close to g_{i-1} for $1 \leq i \leq n$ and so g is homotopic to h .

4. Theorem. *Given $f \in \lim[X, Y]_\alpha$ there is a shape morphism F associated with it.*

Proof. Let $f = \lim\{[f_\alpha]_\alpha: \alpha \in \text{Cov}(Y)\}$ and let $\{\alpha_n: n = 1, 2, \dots\}$ be cofinal in $\text{Cov}(Y)$ with $\alpha_{n+1} < \alpha_n$ for $n = 1, 2, \dots$. Let $n \geq 1$ and suppose we have chosen, using §3, $\gamma_n \in \text{Cov}(\nu(\alpha_n))$ so that γ_n -homotopic continuous functions from X to $\nu(\alpha_n)$ must be homotopic and $\gamma_n^* < \text{st}(\alpha_n)$. Suppose further that, using §2, we have an $\alpha'_n \in \text{Cov}(Y)$ and a continuous function g_n from X to $\nu(\alpha_n)$ so that $\alpha'_n < \alpha_n$, $\alpha'_n < \phi_{\alpha_n}^{-1}(\gamma_n)$ and g_n is

γ_n -homotopic to $\phi_{\alpha_n} f_{\alpha'_n}$. Choose $\gamma_{n+1} \in \text{Cov}(\nu(\alpha_{n+1}))$ to satisfy the condition of §3, and so that $\gamma_{n+1} < \phi_{\alpha_n \alpha_{n+1}}^{-1}(\gamma_n)$ and $\gamma_{n+1} < \text{st}(\alpha_{n+1})$. By §2, choose $\alpha'_{n+1} \in \text{Cov}(Y)$ and g_{n+1} a continuous function from X to $\nu(\alpha_{n+1})$ satisfying the conditions: $\alpha'_{n+1} < \phi_{\alpha_n \alpha_{n+1}}^{-1}(\gamma_{n+1})$, α'_n and α_{n+1} ; $\phi_{\alpha_n} f_{\alpha'_{n+1}}$ is α_n -homotopic to $\phi_{\alpha_n \alpha_{n+1}} \phi_{\alpha_{n+1}} f_{\alpha'_{n+1}}$; and g_{n+1} is γ_{n+1} -homotopic to $\phi_{\alpha_{n+1}} f_{\alpha'_{n+1}}$. Then we have

$$\phi_{\alpha_n \alpha_{n+1}} g_{n+1} \quad \gamma_n\text{-homotopic to} \quad \phi_{\alpha_n \alpha_{n+1}} \phi_{\alpha_{n+1}} f_{\alpha'_{n+1}}$$

(since $\gamma_{n+1} < \phi_{\alpha_n \alpha_{n+1}}^{-1}(\gamma_n)$),

$$\phi_{\alpha_n \alpha_{n+1}} \phi_{\alpha_{n+1}} f_{\alpha'_{n+1}} \quad \gamma_n\text{-homotopic to} \quad \phi_{\alpha_n} f_{\alpha'_{n+1}},$$

$$\phi_{\alpha_n} f_{\alpha'_{n+1}} \quad \gamma_n\text{-homotopic to} \quad \phi_{\alpha_n} f_{\alpha'_n}$$

(since $\alpha'_{n+1} < \alpha'_n < \phi_{\alpha_n}^{-1}(\gamma_n)$), and

$$\phi_{\alpha_n} f_{\alpha'_n} \quad \gamma_n\text{-homotopic to} \quad g_n.$$

Therefore, by choosing g_n inductively in this manner we have $\phi_{\alpha_n \alpha_{n+1}} g_{n+1}$ homotopic to g_n and $g_n \text{st}(\alpha_n)$ -homotopic to $\phi_{\alpha_n} f_{\alpha'_n}$ for $n = 1, 2, \dots$. We can then define a shape morphism F , associated with f , by setting $F([\phi_\alpha]) = [\phi_{\alpha \alpha_n} g_n]$ for $\alpha_n < \alpha \in \text{Cov}(Y)$.

5. Theorem. *Given a shape morphism F from X to Y , there is an $f \in \lim[X, Y]_\alpha$ associated with it.*

Proof. Let $\alpha = \{U_1, \dots, U_n\} \in \text{Cov}(Y)$ and for $p \in \nu(\alpha)$ let $\psi_\alpha(p) \in \bigcap \{U_i : p \in \text{st}(U_i)\}$. Then $p \in \text{st}(U_i)$ implies $\phi_\alpha \psi_\alpha(p) \in \text{st}(U_i)$. Therefore $\phi_\alpha \psi_\alpha$ is $\text{st}(\alpha)$ -homotopic to the identity on $\nu(\alpha)$. Let $g_\alpha \in F([\phi_\alpha])$ and $f_\alpha = \psi_\alpha g_\alpha$. Then f_α is α -continuous and $\phi_\alpha f_\alpha$ is $\text{st}(\alpha)$ -homotopic to g_α . Suppose $\alpha' = \{V_1, \dots, V_m\}$ refines α . Then $p \in \text{st}(V_j)$ implies $\psi_{\alpha'}(p)$ and $\psi_\alpha \phi_{\alpha \alpha'}(p)$ are elements of $\phi_{\alpha \alpha'}(V_j)$. It follows that $\psi_{\alpha'}$ and $\psi_\alpha \phi_{\alpha \alpha'}$ are α -homotopic. This implies $f_{\alpha'} = \psi_{\alpha'} g_{\alpha'}$ is α -homotopic to $\psi_\alpha \phi_{\alpha \alpha'} g_{\alpha'}$ which is α -homotopic to $\psi_\alpha g_\alpha$. So $f_{\alpha'}$ is α -homotopic to f_α . $f = \lim \{f_\alpha\}_\alpha \in \lim[X, Y]_\alpha$, and f is associated with F .

6. Theorem. *If F and G are shape morphisms from X to Y associated respectively with f and $g \in \lim[X, Y]_\alpha$, then $F = G$ if and only if $f = g$.*

Proof. Suppose $f = \lim \{[f_\alpha]_\alpha: \alpha \in \text{Cov}(Y)\}$, $g = \lim \{[g_\alpha]_\alpha: \alpha \in \text{Cov}(Y)\}$, $F_\alpha \in F([\phi_\alpha])$, and $G_\alpha \in G([\phi_\alpha])$, $\alpha \in \text{Cov}(Y)$. Assume first that $F = G$. Then for each $\alpha \in \text{Cov}(Y)$, $\phi_\alpha f_\alpha$ and $\phi_\alpha g_\alpha$ are both $\text{st}(\alpha)$ -homotopic to F_α so $[f_\alpha]_\alpha = [g_\alpha]_\alpha$ and $f = g$. Conversely, assume $f = g$. Then, for $\alpha \in \text{Cov}(Y)$, by §3 there is a $\beta \in \text{Cov}(\nu(\alpha))$ such that any two β -homotopic continuous functions from another space are homotopic. By §2 there is an $\alpha' \in \text{Cov}(Y)$ and a map h from $\nu(\alpha')$ to $\nu(\alpha)$ such that $\text{st}(\alpha') < h^{-1}(\beta)$ and ϕ_α and $h\phi_{\alpha'}$ are β -homotopic hence homotopic. Now $\phi_{\alpha'}/_{\alpha'}$ is $\text{st}(\alpha')$ -homotopic to $F_{\alpha'}$ and to $G_{\alpha'}$. Therefore $hF_{\alpha'}$ is homotopic to $hG_{\alpha'}$. But $hF_{\alpha'}$ is homotopic to F_α and $hG_{\alpha'}$ is homotopic to G_α . Hence $F = G$.

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