PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 1, January 1975

GENERATING FUNCTIONS FOR A SPECIAL CLASS OF PERMUTATIONS

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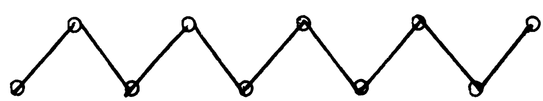
ABSTRACT. The paper contains a simple, direct derivation of the generating functions for a class of permutations generalizing the up-down permutations.

1. A permutation $(a_1, a_2, \cdots, a_n)$ of $Z_n = \{1, 2, \cdots, n\}$ is called an *up-down* permutation if

$$a_1 < a_2,\ a_2 > a_3,\ a_3 < a_4,\ a_4 > a_5, \cdots. \tag{1.1}$$

This can be represented graphically by

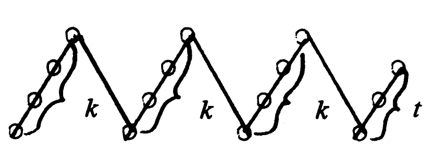


Let $A(n)$ denote the number of up-down permutations of Z_n. It is well known (see for example [3, pp. 105–112]) that

$$\sum_{n=0}^{\infty} A(n)\frac{x^n}{n!} = \sec x + \tan x \qquad (A(0) = A(1) = 1). \tag{1.2}$$

This result can be generalized in the following way. Let k, t be fixed integers, $k \geq 2$, $t \geq 0$. Consider permutations of Z_{kn+t} of the following type:

(1.3)



Each incline (except possibly the last) contains k nodes; the last contains t nodes. In the above illustration, $n = 3$. For brevity we may call the permutations of the form (1.3), with arbitrary n, (k, t)-permutations.

Received by the editors March 28, 1973.
AMS (MOS) subject classifications (1970). Primary 05A15.
[1] Supported in part by NSF grant GP-17031.

Let $A_{k,t}(nk+t)$ denote the number of such permutations of Z_{kn+t}. It is convenient to define

$$(1.4)\qquad \begin{aligned} &A_{k,t}(t)=1,\\ &A_{k,t}(s)=0 \qquad (0\le s<t). \end{aligned}$$

The writer [2] has proved by specializing a general result that

$$(1.5)\qquad \sum_{n=0}^{\infty} A_{k,0}(kn)\frac{x^{kn}}{(kn)!}=\frac{1}{\phi_{k,0}(x)},$$

$$(1.6)\qquad \sum_{n=0}^{\infty} A_{k,t}(kn+t)\frac{x^{kn+t}}{(kn+t)!}=\frac{\phi_{k,t}(x)}{\phi_{k,0}(x)} \qquad (t\ge 1),$$

where

$$(1.7)\qquad \phi_{k,t}(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{kn+t}}{(kn+t)!} \qquad (t=0,1,2,\ldots).$$

The functions $\phi_{k,t}(x)$ defined by (1.7) are the so-called Olivier functions [4]. For some arithmetic properties of the Olivier functions see [1].

For $k=2$, it is clear that (1.5) and (1.6) reduce to (1.2).

The object of the present note is to give a simple self-contained proof of (1.5) and (1.6).

2. We first set up a recurrence for the enumerant $A_{k,t}(kn+t)$. To do this, we consider the effect of removing the element $kn+t$ from a typical (k,t)-permutation of Z_{kn+t}. Unless this element is at the extreme right, the given permutation breaks into two pieces: the left-hand piece is a $(k,k-1)$-permutation, while the right-hand piece is a (k,t)-permutation. If $t=1$, it is clear that the extreme right-hand element cannot be $kn+t$. If $t>1$ and $kn+t$ is in the extreme right-hand position, then clearly its removal leaves a $(k,t-1)$-permutation. We accordingly obtain the following recurrences.

$$(2.1)\qquad A_{k,0}(kn)=\sum_{j=1}^{n}\binom{kn-1}{kj-1}A_{k,k-1}(kj-1)A_{k,0}(k(n-j)),$$

$$(2.2)\qquad A_{k,1}(kn+1)=\sum_{j=1}^{n}\binom{kn}{kj-1}A_{k,k-1}(kj-1)A_{k,1}(k(n-j)+1),$$

$$A_{k,t}(kn+t) = \sum_{j=1}^{n} \binom{kn+t-1}{kj-1} A_{k,k-1}(kj-1)A_{k,t}(k(n-j)+t) \tag{2.3}$$
$$+ A_{k,t-1}(kn+t-1) \qquad (t > 1).$$

Now put

$$F_{k,t}(x) = \sum_{n=0}^{\infty} A_{k,t}(kn+t) \frac{x^{kn+t}}{(kn+t)!} \qquad (t = 0, 1, 2, \ldots). \tag{2.4}$$

For $t = 0$, it follows from (2.1) that

$$F'_{k,0}(x) = \sum_{n=1}^{\infty} A_{k,0}(kn) \frac{x^{kn-1}}{(kn-1)!}$$
$$= \sum_{n=1}^{\infty} \frac{x^{kn-1}}{(kn-1)!} \sum_{j=1}^{n} \binom{kn-1}{kj-1} A_{k,k-1}(kj-1)A_{k,0}(k(n-j))$$
$$= \sum_{j=1}^{\infty} A_{k,k-1}(kj-1) \frac{x^{kj-1}}{(kj-1)!} \sum_{n=0}^{\infty} A_{k,0}(kn) \frac{x^{kn}}{(kn)!}.$$

Hence we get

$$F'_{k,0}(x) = F_{k,k-1}(x)F_{k,0}(x). \tag{2.5}$$

Next, for $t = 1$, it follows from (2.2) that

$$F'_{k,1}(x) = \sum_{n=0}^{\infty} A_{k,1}(kn+1) \frac{x^{kn}}{(kn)!}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{x^{kn}}{(kn)!} \sum_{j=1}^{n} \binom{kn}{kj-1} A_{k,k-1}(kj-1)A_{k,1}(k(n-j)+1)$$
$$= 1 + \sum_{j=1}^{\infty} A_{k,k-1}(kj-1) \frac{x^{kj-1}}{(kj-1)!} \sum_{n=0}^{\infty} A_{k,1}(kn+1) \frac{x^{kn+1}}{(kn+1)!}.$$

Thus

$$F'_{k,1}(x) = 1 + F_{k,k-1}(x)F_{k,1}(x). \tag{2.6}$$

Finally, for $t > 1$, we have

$$F'_{k,t}(x) = \sum_{n=0}^{\infty} A_{k,t}(kn+t)\frac{x^{kn+t-1}}{(kn+t-1)!}$$

$$= A_{k,t}(t)\frac{x^{t-1}}{(t-1)!}$$

$$+ \sum_{n=1}^{\infty} \frac{x^{kn+t-1}}{(kn+t-1)!}\left\{\sum_{j=1}^{n}\binom{kn+t-1}{kj-1}A_{k,k-1}(kj-1)A_{k,t}(k(n-j)+t)\right.$$

$$\left. + A_{k,t-1}(kn+t-1)\right\}$$

$$= \sum_{n=0}^{\infty} A_{k,t-1}(kn+t-1)\frac{x^{kn+t-1}}{(kn+t-1)!}$$

$$+ \sum_{j=1}^{\infty} A_{k,k-1}(kj-1)\frac{x^{kj-1}}{(kj-1)!}\sum_{n=0}^{\infty} A_{k,t}(kn+t)\frac{x^{kn+t}}{(kn+t)!}.$$

Therefore

$$F'_{k,t}(x) = F_{k,t-1}(x) + F_{k,k-1}(x)F_{k,t}(x) \qquad (t > 1). \tag{2.7}$$

3. In order to solve the system (2.5), (2.6), (2.7) for the $F_{k,t}(x)$ we put

$$F_{k,0}(x) = 1/\psi_{k,0}(x), \tag{3.1}$$

$$F_{k,t}(x) = \psi_{k,t}(x)/\psi_{k,0}(x) \qquad (t > 0). \tag{3.2}$$

Since the series defining $F_{k,0}(x)$ has constant term equal to 1, it is clear that $\psi_{k,0}(x)$, $\psi_{k,t}(x)$ are well defined.

Substituting from (3.1) and (3.2) in (2.5), we get

$$-\frac{\psi'_{k,0}(x)}{\psi^2_{k,0}(x)} = \frac{\psi_{k,k-1}(x)}{\psi_{k,0}(x)}\frac{1}{\psi_{k,0}(x)},$$

so that

$$\psi'_{k,0}(x) = -\psi_{k,k-1}(x). \tag{3.3}$$

Similarly, it follows from (2.6) that

$$\frac{\psi_{k,0}(x)\psi'_{k,1}(x) - \psi'_{k,0}(x)\psi_{k,1}(x)}{\psi^2_{k,0}(x)} = 1 + \frac{\psi_{k,k-1}(x)\psi_{k,1}(x)}{\psi^2_{k,0}(x)}.$$

Making use of (3.3), this reduces to

$$\psi'_{k,1}(x) = \psi_{k,0}(x). \tag{3.4}$$

In exactly the same way, it follows from (2.7) and (3.3) that

$$\psi'_{k,t}(x) = \psi_{k,t-1}(x) \qquad (t > 1). \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\psi'_{k,t}(x) = \psi_{k,t-1}(x) \qquad (t \geq 1). \tag{3.6}$$

It follows at once from (3.3) and (3.6) that

$$d^k\psi_{k,0}(x)/dx^k = -\psi_{k,0}(x). \tag{3.7}$$

It is clear from (3.1) and (2.4) that

$$\psi_{k,0}(x) = \sum_{n=0}^{\infty} a_n \frac{x^{kn}}{(kn)!} \qquad (a_0 = 1).$$

Thus

$$\frac{d^k}{dx^k}\psi_{k,0}(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^{kn}}{(kn)!}.$$

Hence by (3.7), $a_{n+1} = -a_n$, so that $a_n = (-1)^n$. It follows that

$$\psi_{k,0}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{kn}}{(kn)!}. \tag{3.8}$$

Applying (3.3) we get

$$\psi_{k,k-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{kn+k-1}}{(kn+k-1)!}. \tag{3.9}$$

Repeated application of (3.6) now gives

$$\psi_{k,t}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{kn+t}}{(kn+t)!} \qquad (0 \leq t < k). \tag{3.10}$$

To remove the restriction on t in (3.10), we note first that, by (3.2),

$$\psi_{k,t}(x) = \sum_{n=0}^{\infty} a_{n,t} \frac{x^{kn+t}}{(kn+t)!} \qquad (t \geq 0).$$

In view of (3.6) we have

$$a_{n,t} = a_{n,t-1} \qquad (t \geq 1). \tag{3.11}$$

Since, in particular, $a_{n,k-1} = (-1)^n$, it follows from (3.11) that $a_{n,t} = (-1)^n$ $(t \geq k)$. Therefore

$$\psi_{k,t}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{kn+t}}{(kn+t)!} \qquad (t \geq 0). \tag{3.12}$$

Comparing (3.12) with (1.7) we get

$$\psi_{k,t}(x) = \phi_{k,t}(x) \qquad (t \geq 0).$$

This evidently completes the proof of (1.5) and (1.6).

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