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COUNTING PATTERNS WITH A GIVEN AUTOMORPHISM GROUP

DENNIS E. WHITE[1]

ABSTRACT. A formula, analogous to the classical Burnside lemma, is developed which counts orbit representatives from a set under a group action with a given stabilizer subgroup conjugate class. This formula is applied in a manner analogous to a proof of Pólya's theorem to obtain an enumeration of patterns with a given automorphism group.

1. Let S be a finite set and G a finite group acting on S. Let Δ be a system of orbit representatives for G acting on S. The following theorem is well known:

Theorem 1 (Burnside [1]). *For any function ω defined on S satisfying $\omega(\sigma s) = \omega(s)$ for all $\sigma \in G$, for all $s \in S$, we have*

$$\sum_{s \in \Delta} \omega(s) = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{s \in S} \omega(s) \chi(\sigma s = s)$$

where

$$\chi(\textit{statement}) = \begin{cases} 1 & \textit{if statement is true}, \\ 0 & \textit{otherwise}. \end{cases}$$

For $s \in S$ let $G_s = \{\sigma \in G: \sigma s = s\}$ be the stabilizer subgroup of G at s. Let $G_1, G_2, \cdots, G_N$ be a complete set of nonconjugate subgroups of G, ordered such that $|G_1| \geq \cdots \geq |G_N|$. For any two subgroups $H, K \subset G$ we define

$$M_K(H) = \frac{1}{|K|} \sum_{\sigma \in G} \chi(\sigma H \sigma^{-1} \subset K).$$

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$M_K(H)$ is sometimes called the *mark* of K at H. The matrix $M = (M_{G_j}(G_i))$ is triangular and $M_{G_i}(G_i) \geq 1$ so that we can define $B = M^{-1}$, $B = (b_{ij})$. We also note that $M_K(H)$ is constant on conjugate subgroups of G.

In this paper we show the following result:

Theorem 2. *For any function ω defined on S satisfying $\omega(\sigma s) = \omega(s)$ for all $\sigma \in G$, for all $s \in S$, we have*

$$\sum_{s \in \Delta} \omega(s)\chi(G_s \sim G_i) = \sum_{j=1}^{N} b_{ij} \sum_{s \in S} \omega(s)\chi(G_j s = s),$$

where $G_s \sim G_i$ means G_s conjugate to G_i and $G_j s = s$ means s is fixed by all of G_j.

In an elegant paper [2], DeBruijn showed that Pólya's counting theorem [5] can be obtained from Theorem 1 upon letting $S = R^D$, where R^D is the set of functions from the finite set $D = \{1, 2, \cdots, |D|\}$ to the finite set $R = \{1, 2, \cdots, |R|\}$, letting G act on D and hence on R^D by setting $\sigma f(d) = f(\sigma^{-1} d)$, and setting $\omega(f) = \Pi_{d \in D} x_{f(d)}$, where $x_1, x_2, \cdots$ are indeterminate. If we use the same approach, starting from Theorem 2 instead of Theorem 1, with no additional difficulty we obtain a more refined version of Pólya's theorem.

Let $Q_i(x_1, x_2, \cdots)$ denote the *pattern inventory* for patterns whose automorphism group is conjugate to G_i:

$$Q_i(x_1, x_2, \cdots) = \sum_{f \in \Delta} \omega(f)\chi(G_f \sim G_i).$$

Let $P_i(y_1, y_2, \cdots)$ denote the *orbit index monomial*:

$$P_i(y_1, y_2, \cdots) = \prod_{d \in D} y_d^{q_{G_i}(d)},$$

where $q_{G_i}(d) =$ the number of orbits of G_i acting on D of length d, and $y_1, y_2, \cdots$ are indeterminates. Then we have

Theorem 3.

$$Q_i(x_1, x_2, \cdots) = \sum_{j=1}^{N} b_{ij} P_j(y_1, y_2, \cdots)$$

where we substitute $\Sigma_{r \in R} x_r^i$ for y_i.

This result was proved independently by Stockmeyer [8]. However, he obtained it only as a by-product of elaborate Möbius function techniques.

We show here that Theorem 3 can be derived by simple algebraic manipulations.

We were led to this result by considering the general isomorph rejection problem in a multilinear setting [9], [10]. In this setting, besides Theorem 3, we have also derived from Theorem 2 a whole variety of results counting patterns with a given automorphism group. In particular, for example, we may let G act on R and D or let G act on D and H act on R. Or we may extend S to be a cartesian product of finite function spaces, G acting on each of them. Or we may observe that a theorem of Foulkes [3] is nothing more than Theorem 2 applied to a special function space.

2. We shall first prove Theorem 2. The weight function ω in this theorem is commonly thought of as a function from S into an algebra, usually the algebra of polynomials.

Proof of Theorem 2. Note that for any subgroup $H \subset G$, $\Sigma_{\sigma\epsilon G}\, \chi(\sigma H\sigma^{-1} \subset G_s)$ is constant on orbits of S, so if we denote the orbit of s by O_s and recall that $|G| = |G_s|\,|O_s|$ we have

$$\sum_{i=1}^{N} M_{G_i}(H) \sum_{s\epsilon\Delta} \omega(s)\chi(G_s \sim G_i) = \sum_{s\epsilon\Delta} \frac{\omega(s)}{|G_s|} \sum_{\tau\epsilon G} \chi(\tau H\tau^{-1} \subset G_s)$$

$$= \sum_{s\epsilon S} \frac{\omega(s)}{|G_s|\,|O_s|} \sum_{\tau\epsilon G} \chi(\tau H\tau^{-1} \subset G_s)$$

$$= \frac{1}{|G|} \sum_{\tau\epsilon G} \sum_{s\epsilon S} \omega(\tau s)\chi(H \subset G_{\tau s}) = \sum_{s\epsilon S} \omega(s)\chi(H \subset G_s).$$

Inverting M gives our result.

We shall now use Theorem 2 to prove Theorem 3. The similarities between this proof and the proof of Pólya's theorem in [2] are obvious.

Proof of Theorem 3. Note that

$$Q_i(x_1, x_2, \cdots) = \sum_{j=1}^{N} b_{ij} \sum_{f\epsilon R^D} \omega(f)\chi(G_j f = f).$$

But $G_j f = f$ means $\sigma f = f$ for all $\sigma \in G_j$, or $f(d) = f(\sigma^{-1}d)$ for all $d \in D$, for all $\sigma \in G_j$. Thus, f must be restricted to be constant on the orbits of G_j acting on D. We can then define f such that $G_j f = f$ by defining it on each orbit. Thus,

$$\sum_{f\epsilon R^D} \omega(f)\chi(G_j f = f) = \sum_{f\epsilon R^{\mathrm{Orb}(G_j:D)}} \prod_{A\epsilon \mathrm{Orb}(G_j:D)} x_{f(A)}^{|A|}$$

where $\mathrm{Orb}(G_j: D)$ is the set of orbits of G_j acting on D. Using the familiar sum-product interchange gives

$$\sum_{f \in R^D} \omega(f)\chi(G_j f = f) = \prod_{A \in \mathrm{Orb}(G_j:D)} \sum_{r \in R} x_r^{|A|}$$

$$= \prod_{d \in D} \Big(\sum_{r \in R} x_r^d\Big)^{q_{G_j}^{(d)}} = P_j\Big(\sum_{r \in R} x_r, \sum_{r \in R} x_r^2, \cdots\Big). \quad \text{Q.E.D.}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92037

Current address: School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455