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ABSOLUTE SUMMABILITY MATRICES THAT ARE STRONGER THAN THE IDENTITY MAPPING

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ABSTRACT. The main result gives a simple column-sum property which implies that the matrix A maps l_A properly into l^1, i.e., $l^1 \subsetneq A^{-1}[l^1]$. Also, the means of Nörlund, Euler-Knopp, Taylor, and Hausdorff are investigated as mappings of l^1 into itself.

1. Introduction. Let A be an infinite matrix defining a sequence to sequence summability transformation by $(Ax)_n = \Sigma_{k\geq 1} a_{nk}x_k$. The inverse image of l^1 under A is denoted by l_A; and A is called an l-l matrix provided that $l^1 \subseteq l_A$. In [7] Knopp and Lorentz proved that A is an l-l matrix if and only if there is a number M such that for each k,

$$\sum_{n\geq 1} |a_{nk}| \leq M. \tag{1}$$

Also, A is sum-preserving if for each x in l^1, $\Sigma_{n\geq 1} (Ax)_n = \Sigma_{k\geq 1} x_k$. The l-l matrix A is sum-preserving if and only if for each k,

$$\sum_{n\geq 1} a_{nk} = 1. \tag{2}$$

In [2] Agnew gave a simple sufficient condition that A maps a nonconvergent sequence into a convergent one. The principal result of this paper is an analogue of this condition for l-l matrices; i.e., we shall give an explicit property of the terms $\{a_{nk}\}$ that implies $l \subsetneq l_A$. In the final section, we investigate the absolute summability properties of some well-known matrix methods.

2. The main theorem. Following Agnew, we might conjecture that $\lim_{n,k} a_{nk} = 0$ implies $l_A \neq l$. (The double limit is taken in the Pringsheim

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sense: $|a_{nk}| < \epsilon$ whenever $n > N$ and $k > N$.) This conjecture is reinforced by the observation that it is true in case A is lower triangular with $a_{nn} \neq 0$; for then A^{-1} is not l-l because $\sup_n |a_{nn}|^{-1} = \infty$. However, the following example shows that, even for lower triangular matrices, this property is not sufficient in general.

Example. If

$$a_{nk} = \begin{cases} 1/k, & \text{if } k(k-1)/2 < n \leq k(k+1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

then A is a lower triangular, sum-preserving l-l matrix for which $\lim_{n,k} a_{nk} = 0$, but $l_A = l$.

If Agnew's property is replaced by $\lim_k \sum_{n\geq 1} |a_{nk}| = 0$, then it is easy to construct an unbounded sequence x such that $\sum_n |(Ax)_n|$ converges. Indeed, it is sufficient that only a subsequence of the column sums tends to zero, so we can state the following result (cf. [4 Proposition]).

Proposition. *If A is a matrix such that*

$$\liminf_k \sum_{n\geq 1} |a_{nk}| = 0, \tag{3}$$

then $l_A \neq l$.

The simplicity of condition (3) is offset by the fact that it precludes (2), and therefore, no sum-preserving matrix can satisfy it. It is, therefore, necessary to weaken (3), which leads us to the main result.

Theorem 1. *If A is an l-l matrix for which there exists an integer m such that*

$$\liminf_k \sum_{n\geq m} |a_{nk}| = 0, \tag{4}$$

then $l \subsetneqq l_A$.

Proof. The hypothesis (4) implies the existence of an increasing integer sequence $\{k(i)\}$ such that for each i, $\sum_{n\geq m} |a_{n,k(i)}| < 1/i$. If x is chosen so that $|x_{k(i)}| \leq 1/i$, and $x_k = 0$ when $k \neq k(i)$, then

$$\sum_{n\geq m} |(Ax)_n| \leq \sum_{n\geq m} \sum_{i\geq 1} |a_{n,k(i)}| i^{-1}$$
$$= \sum_{i\geq 1} \sum_{n\geq m} |a_{n,k(i)}| i^{-1} < \sum_{i\geq 1} i^{-2}.$$

Hence, Ax is in l^1 *provided that x is in the domain of A.* The difficulty is that we must ensure the convergence of $\Sigma_i a_{n,k(i)} x_{k(i)}$ when $n < m$. To achieve this we must choose a subsequence $\{\kappa(i)\}$ of $\{k(i)\}$ so that for each $n < m$, $\Sigma_i a_{n,\kappa(i)} x_{\kappa(i)}$ converges, while $|x_{\kappa(i)}| = 1/i$. Since (1) implies that the row sequences are bounded, the proof will be completed by the following lemma.

Lemma. *If, for each n less than m, $\{a_{nk}\}_{k=1}^{\infty}$ is a bounded sequence, then there exists a sequence x that is not in l^1 such that $\Sigma_k a_{nk} x_k$ converges for each n less than m.*

Proof. Let M_n denote $\limsup_k |a_{nk}|$, and assume—without loss of generality—that $M_1 \geq M_2 \geq \cdots \geq M_{m-1} \geq 0$. We may also assume that the a_{nk}'s are real, for otherwise we could treat $\{\mathrm{Re}\,(a_{nk})\}_{k=1}^{\infty}$ and $\{\mathrm{Im}\,(a_{nk})\}_{k=1}^{\infty}$ separately and have $2(m-1)$ bounded sequences.

First, suppose $M_{m-1} > 0$. Choose a subsequence $\{a_{1,k(i)}\}_i$ of $\{a_{1,k}\}_k$ such that the terms are either all positive or all negative and for each i,

$$\frac{i+1}{i+2} M_1 \leq |a_{1,k(i)}| \leq \frac{i^2}{i^2-1} M_1.$$

Now choose a subsequence $\{k^*(i)\}$ of $\{k(i)\}$ so that all of the terms $\{a_{2,k^*(i)}\}$ are of the same sign and for each i,

$$\frac{i+1}{i+2} M_2 \leq |a_{2,k^*(i)}| \leq \frac{i^2}{i^2-1} M_2.$$

Choose successive subsequences so that after $m-1$ selections, we have a subsequence $\{\kappa(i)\}$ of the positive integers such that if $n < m$, then $\{a_{n,\kappa(i)}\}_i$ are all of the same sign, and for each i,

$$\frac{i+1}{i+2} M_n \leq |a_{n,\kappa(i)}| \leq \frac{i^2}{i^2-1} M_n. \tag{5}$$

Now define x by

$$x_k = \begin{cases} (-1)^i/i, & \text{if } k = \kappa(i) \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Sigma_k a_{nk} x_k = \Sigma_i a_{n,\kappa(i)} (-1)^i/i$. This is obviously an alternating series whose general term tends to 0. Also, from (5) we have

$$\frac{1}{i} |a_{n,\kappa(i)}| \leq \frac{i}{i^2-1} M_n \leq \frac{1}{i-1} \frac{i}{i+1} M_{n-1} \leq \frac{1}{i-1} |a_{n,\kappa(i)}|.$$

Hence, by the familiar alternating series test, $\Sigma_k a_{nk}x_k$ is convergent.

In case $M_{m-1} = 0$, the preceding construction will yield an x for which $\Sigma_k a_{nk}x_k$ converges when $M_n > 0$. Then remaining sequences $\{a_{nk}\}$ for which $M_n = 0$ are null sequences, and the selection of subsequences for which $\Sigma_k a_{nk}x_k$ converges is straightforward. Hence, $\Sigma_k a_{nk}x_k$ converges for every n less than m, which completes the proof of the Lemma and the Theorem.

Although it is possible for a sum-preserving l-l matrix to satisfy (4), it is easy to see that no lower triangular matrix can satisfy both (2) and (4). Indeed, if A is lower triangular, then (4) implies (3). This leads us to conjecture that perhaps a weaker condition, such as $\liminf_k |\Sigma_{n\geq m} a_{nk}| = 0$, might be sufficient to imply $l_A \neq l^1$. However, if

$$a_{nk} = \begin{cases} 1, & \text{if } n = 2k-1, \\ -1, & \text{if } n = 2k, \\ 0, & \text{otherwise,} \end{cases}$$

then $\Sigma_{n\geq m} a_{nk} = 0$ if $k \neq m/2$, but clearly $l_A = l$.

3. Absolute summability properties of some classical methods. Since many of the classical means are given by lower triangular matrices with nonzero diagonal terms, we can use the following observation in place of (3) for such matrices: $l_A \subsetneqq l^1$ if and only if A satisfies (1) but A^{-1} does not. Note that (3) implies that A^{-1} does not satisfy (1) because $\sup_n |a_{nn}|^{-1} = \infty$.

The Nörlund mean N_p is given by $N_p[n, k] = p_{n-k}/P_n$ if $k \leq n$, and $N_p[n, k] = 0$ if $k > n$, where p is a nonnegative number sequence such that $p_0 > 0$ and $P_n \equiv \Sigma_{k=0}^n p_k$.

Theorem 2. *The Nörlund mean N_p is an l-l matrix if and only if p is in l^1.*

Proof. If p is in l^1, then for each k

$$\sum_{n=0}^{\infty} N_p[n, k] = \sum_{n=k}^{\infty} \frac{p_{n-k}}{P_n} \leq P_0^{-1} \sum_{n=k}^{\infty} p_{n-k} = p_0^{-1} \sum_{i=0}^{\infty} p_i;$$

hence, N_p satisfies (1). Conversely, if p is not in l^1, then by a result of Abel (see footnote in [6, p. 45]), $\lim_n \{1/P_n\} = 0$ while $\Sigma_n p_n/P_n$ diverges. Thus N_p is not an l-l matrix.

From Theorem 2, we see that if $l^1 \subseteq l_{N_p}$, then $\liminf_k \{p_k/P_k\} > 0$, so property (4) does not hold. However, we can prove a Mercerian-type theorem.

For, if $\frac{1}{2} < r < 1$ and $p_0 \geq rP_n$, then $N_p[n, n] \geq r$ (for every n). Therefore, by [5, Theorem 6], we conclude that $l_{N_p} = l^1$. This proves the following assertion.

Theorem 3. *If* $p_0 > 2 \sum_{k\geq 1} p_k$, *then* $l_{N_p} = l^1$.

A particular example of Theorem 3 is seen in case p is a geometric sequence: more precisely, if $p_{k+1} < p_0 3^{-k-1}$, then $l_{N_p} = l^1$.

The Euler-Knopp means ([1], [6], and [9]) are given by

$$E_r[n, k] = \begin{cases} \binom{n}{k} r^k (1 - r)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

A straightforward application of the Maclaurin series expansion of $(1 - z)^{k+1}$ shows that each column sum of E_r converges absolutely to $1/r$ provided that $0 < r \leq 1$. If $0 < r < 1$, then $\lim_n E_r[n, n] = 0$, so E_r^{-1} is not an *l-l* matrix. We summarize this as follows:

Theorem 4. *The Euler-Knopp mean* rE_r *is a sum-preserving l-l matrix for which* $l_{E_r} \neq l^1$ *if and only if* $0 < r < 1$.

The Taylor methods ([3], [8], and [9]), which are given by

$$T_r[n, k] = \begin{cases} 0, & \text{if } k < n, \\ \binom{n}{k} r^{k-n} (1 - r)^{n+1}, & \text{if } k \geq n, \end{cases}$$

are related to the Euler-Knopp means by a transpose relationship. More precisely, if E^* denotes the transpose of E, then $T_r = (1 - r)E^*_{1-r}$. It follows that T_r is an *l-l* matrix for precisely those r's for which E_r maps bounded sequences into bounded sequences, viz., $0 \leq r \leq 1$. We note that (4) is not satisfied by T_r when $0 \leq r < 1$; for, each column sum is $1 - r$, and since the first m rows are null sequences we must have $\sum_{n\geq m} a_{nk} \geq (1 - r)/2$ for sufficiently large k.

The Hausdorff means ([6] and [9]) can be defined by

$$H_\phi[n, k] = \int_0^t E_t[n, k]\, d\phi(t),$$

where E_t is the Euler-Knopp mean and $\int_0^1 |d\phi| < \infty$. The quasi-Hausdorff mean H^*_ϕ is simply the transpose of H_ϕ. Therefore, H_ϕ is an *l-l* method if and only if H^*_ϕ is a bounded operator, and Hardy [6, pp. 278–279] characterizes this by $\int_0^1 |d\phi(t)|/t < \infty$. Furthermore, the column sums are

$$\sum_{n=0}^{\infty} H_\phi[n, k] = \sum_{k=0}^{\infty} H^*_\phi[n, k] = \int_0^1 \frac{d\phi(t)}{t}.$$

Thus we may state the following result.

Theorem 5. *The Hausdorff matrix* H_ϕ *generated by the mass function* ϕ *is an l-l matrix if and only if* $\int_0^1 |d\phi(t)|t^{-1}$ *converges. Moreover,* H_ϕ *is sum-preserving if and only if* $\int_0^1 t^{-1}\, d\phi(t) = 1$.

The corresponding theorem for H^*_ϕ can be stated without proof since it depends upon only the regularity conditions for H_ϕ [6, pp. 256–258].

Theorem 6. *The quasi-Hausdorff matrix* H^*_ϕ *is an l-l method if and only if* ϕ *is a function of bounded variation on* $[0, 1]$. *Moreover,* H^*_ϕ *is sum-preserving if and only if* $\phi(1) - \phi(0) = 1$.

Note that in Theorem 6 it is not required that $\phi(0+) = \phi(0)$, so ϕ need not be a "regular" mass function. Since $\phi(0+) - \phi(0) = \lim_n H_\phi[n, 0] = \lim_k H^*_\phi[0, k]$, it might seem possible that H^*_ϕ is an *l-l* method and satisfies (4). Unfortunately this cannot be the case, because if $k > 0$, then

$$\lim_n H_\phi[n, k] = 0,$$

and

$$\sum_{n \ge m} |H^*_\phi[n, k]| = \sum_{n=0}^{\infty} |H^*_\phi[n, k]| - \sum_{n=0}^{m-1} |H^*_\phi[n, k]|$$

$$= \int_0^1 |d\phi| - |H^*_\phi[0, k]| - \sum_{n=1}^{m-1} |H^*_\phi[n, k]|$$

$$= \frac{1}{2}\int_{0+}^1 |d\phi|$$

for k sufficiently large.

Finally, we remark on the conspicuous absence from our study of the very familiar Cesàro means. The fact is that they are not *l-l* methods. For, if $\alpha > 0$ and $\phi(t) = 1 - (1-t)^\alpha$, then H_ϕ is the Cesàro mean of order α [6, p. 275]. But clearly $\int_0^1 t^{-1} d\phi(t)$ is divergent, so by Theorem 5, C_α is not an *l-l* method.

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