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ON THE SET OF EXTREME POINTS OF A CONVEX BODY

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ABSTRACT. We prove the following: Given a subset X of a compact 0-dimensional metric space Z and an integer $d \geq 3$, there is a homeomorphism of Z into the boundary of a convex body C in E^d mapping X onto the set of extreme points of C if and only if X is a G_δ set with at least $d + 1$ points.

A convex body is any closed bounded convex set with nonempty interior. It is well known that the set of extreme points of a convex body in E^d is a G_δ set with at least $d+1$ points. We may consider as a partial converse to this the fact that for each integer $n \geq d+1$ there is a convex body in E^d with exactly n extreme points. The purpose of this note is to prove a more general converse. We denote the set of extreme points of a convex set C by ext C. Our main result is

Theorem 1. *Let X be a subset of a compact 0-dimensional metric space Z, and let d be an integer with $d \geq 3$. There is a homeomorphism of Z into the boundary of a convex body C in E^d mapping X onto* ext C *if and only if X is a G_δ set with at least $d+1$ points.*

The boundary of any convex body in E^d is homeomorphic to the $(d-1)$-sphere, S^{d-1}. For each positive integer d we define χ_d to be the family of all subsets X of S^{d-1} for which there is a homeomorphism of S^{d-1} onto the boundary of a convex body C in E^d mapping X onto ext C. The family we call χ_3 was first defined in [3] and the question raised of finding a characterization of that family. It is known that each member of χ_d is a G_δ set with at least $d+1$ points and that each closed subset of S^{d-1} with at least $d+1$ points is a member of χ_d. However, no proper open subset of S^{d-1} is in χ_d. Using Theorem 1 we may further state

Corollary 1. *Any G_δ subset of S^2 with at least four points and whose closure is 0-dimensional is a member of χ_3.*

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Corollary 2. *For $d \geq 3$ any subset of S^{d-1} with at least $d+1$ points and whose closure is countable is a member of χ_d.*

These follow easily from results in [4, p. 532] and [2, p. 456], respectively, on extending homeomorphisms.

We proceed to the proof of Theorem 1. The closure, the boundary, and the convex hull of a subset A of E^d will be denoted by $\mathrm{cl}\, A$, $\mathrm{bd}\, A$, and $\mathrm{conv}\, A$, respectively. For any compact subset K of the real line $\mathbf{R}$, $\mathbf{R}\backslash K$ is the disjoint union of countably many open intervals. Let $e(K)$ be the set of endpoints of these intervals; then $e(K)$ is a countable subset of K.

Lemma 1. *Let Y be a compact 0-dimensional subset of $\mathbf{R}$, and X a G_δ subset of Y which contains $e(Y)$. Then there is a continuous function $g: Y \to \mathbf{R}$ such that $X \cap g^{-1}(s)$ is the set of extreme points of* $\mathrm{conv}\, g^{-1}(s)$ *for each $s \in g[Y]$.*

Proof. We construct a sequence $\mathfrak{U}_0, \mathfrak{U}_1, \cdots$ of open (and also closed) coverings of Y by pairwise disjoint sets so that for each $n \geq 0$, $\mathfrak{U}_{n+1}$ is a refinement of $\mathfrak{U}_n$. For each U in $\mathfrak{U}_n$ we shall specify a compact subset $c_n(U)$ of U.

Since X is a G_δ set, $Y\backslash X$ is the countable union of a nested chain $K_1 \subseteq K_2 \subseteq \cdots$ of compact sets. We may choose $\mathfrak{U}_0 = \{Y\}$ and $c_0(Y) = \emptyset$ and suppose that $\mathfrak{U}_{n-1}$ and c_{n-1} have been defined for some $n \geq 1$. To complete the inductive construction, we need only specify for an arbitrary $U \in \mathfrak{U}_{n-1}$ the members $V_0, \cdots, V_k$ of $\mathfrak{U}_n$ which are subsets of U and define $c_n(v_i)$ for $i = 0, \cdots, k$. Since U is compact and 0-dimensional, there is a finite open covering $\mathfrak{V}$ of U by pairwise disjoint sets each of diameter $\leq 1/n$. Let V_0 be the union of those members of $\mathfrak{V}$ which intersect $c_{n-1}(U)$ and let $V_1, \cdots, V_k$ be an enumeration of the remaining sets in $\mathfrak{V}$. Choose $c_n(V_0) = c_{n-1}(U)$. By hypothesis the extreme points a_i, b_i of $\mathrm{conv}\, V_i$ are in X. We define $c_n(V_i) = \{a_i, b_i\} \cup (K_n \cap V_i)$ for $i = 1, \cdots, k$. Hence the extreme points of $\mathrm{conv}\, c_n(V_i)$ are a_i, b_i and these are the only points of $c_n(V_i)$ which lie in X.

Consider each $\mathfrak{U}_n$ to be a topological space with the discrete topology. Then there is a continuous mapping $f_n: \mathfrak{U}_n \to \mathfrak{U}_{n-1}$, $n \geq 1$, defined by $U \subseteq f_n(U)$, and there is a continuous mapping $g_n: Y \to \mathfrak{U}_n$ defined by $y \in g_n(y)$. Thus $\{\mathfrak{U}_n, f_n\}$ is an inverse limit sequence. Let $\mathfrak{U}_\infty$ be the inverse limit space of $\{\mathfrak{U}_n, f_n\}$, and let $g_\infty: Y \to \mathfrak{U}_\infty$ be the continuous mapping induced by $g_n: Y \to \mathfrak{U}_n$. Since Y is compact and $\mathfrak{U}_n$ is discrete for each $n \geq 0$, $\mathfrak{U}_\infty$ is compact and 0-dimensional; hence there is a homeomorphism

$h\colon \mathfrak{U}_\infty \to \mathbf{R}$. We define $g = h \circ g_\infty$. From the construction of g_∞ it is evident that for $x, y \in Y$, $g(x) = g(y)$ if and only if either $x = y$ or $x, y \in c_n(U)$ for some $U \in \mathfrak{U}_n$ and $n \geq 0$. Since each point in $Y \backslash X$ lies in some $c_n(U)$ but is not an extreme point of conv $c_n(U)$, g has the desired property. □

The next lemma may also be proved using inverse limit spaces. However, we omit the details since the method is similar to that used in [1, §§2–15].

Lemma 2. *Let Y be a compact, 0-dimensional metric space and X a countable dense subset of Y. Then there is a homeomorphism $h\colon Y \to \mathbf{R}$ such that $e(h[Y]) = h[X]$.*

Proof of Theorem 1. One direction is immediate. To prove the other direction, suppose X is an infinite G_δ subset of Z. Lemma 2 implies that we may assume cl $X = Y \subseteq \mathbf{R}$ with $e(Y) \subseteq X$. Let $g\colon Y \to \mathbf{R}$ be as in Lemma 1, and define $G\colon Y \to E^3$ by $x \to (x, g(x), g^2(x))$. This is a homeomorphism of Y into the graph of the convex function on E^2 given by $(x, y) \to y^2$. Let $C = \text{conv } G[Y]$, then ext $C = G[X]$. Since we may extend G to a homeomorphism $h\colon Z \to \text{bd } C$, we have established Theorem 1 for $d = 3$.

Assume the theorem is true for $d = n$, and express Z as the disjoint union of closed subsets Z_1, Z_2 so that $X_i = X \cap Z_i$ is infinite, $i = 1, 2$. By assumption there is a convex body C_i in E^n and a homeomorphism $h_i\colon Z_i \to \text{bd } C_i$ with $h_i[X_i] = \text{ext } C_i$. We may consider C_1, C_2 to lie in distinct parallel hyperplanes in E^{n+1}. Let $C = \text{conv}(C_1 \cup C_2)$ and define the homeomorphism $h\colon Z \to \text{bd } C$ by $h/Z_i = h_i$, $i = 1, 2$. Then C is a convex body in E^{n+1} and $h[X] = \text{ext } C$. □

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