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COMMUTATIVE REGULAR RINGS WITHOUT PRIME MODEL EXTENSIONS

D. SARACINO[1] AND V. WEISPFENNING[2]

ABSTRACT. It is known that the theory K of commutative regular rings with identity has a model completion K'. We show that there exists a countable model of K which has no prime extension to a model of K'.

If K and K' are theories in a first order language L, then K' is said to be a model completion of K if K' extends K, every model of K can be embedded in a model of K', and for any model A of K and models B_1, B_2 of K' extending A, we have $\langle B_1 \underline{a}\rangle_{a \in A} \equiv \langle B_2, \underline{a}\rangle_{a \in A}$, i.e. B_1 and B_2 are elementarily equivalent in a language which has constants for the elements of A. If a theory K has a model completion K', then the models of K' can reasonably be regarded as the "algebraically closed" models of K; for example, the theory of algebraically closed fields is the model completion of the theory of fields. It was shown in [3] that the theory K of commutative regular rings with identity (formulated in the usual language L for rings with identity) has a model completion. We recall that a commutative ring R with identity is said to be regular (in the sense of von Neumann) if for any element $a \in R$ there exists $b \in R$ such that $a^2b = a$. (A good reference is Lambek [2].) The model completion K' is given by the following axioms:

(i) the axioms of commutative regular rings with identity;

(ii) an axiom stating that there are no minimal idempotents, i.e.

$$\forall x(x^2 = x \wedge x \neq 0 \rightarrow \exists y(y^2 = y \wedge y \neq 0 \wedge y \neq x \wedge yx = y));$$

(iii) a set of axioms stating that every monic polynomial has a root.

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We will also consider the theory K^* obtained by deleting axiom (ii) from K'.

If B is a model of a theory T, and A is a substructure of B, then B is called a prime model extension (for T) of A, if for every model C of T extending A, there exists an embedding $f: B \to C$ which is the identity on A. Of course if T is model-complete (as in K'), then "embedding" can be replaced by "elementary embedding". In this paper we consider the question of whether over every commutative regular ring A there exists a prime model extension for K'; we also consider the same question for K^*. In both cases the answer is negative.

We begin by recalling some model theoretic preliminaries in the setting of commutative regular rings. Let R be a model of K; we expand L to the language $L(R)$ by adding a new constant symbol $\underline{a}$ for every element $a \in R$. The diagram $D(R)$ of R is the set of all polynomial equations and inequations involving the constants $\underline{a}$ which hold in R when $\underline{a}$ is interpreted as a. Every model of the theory $K' \cup D(R)$ is (up to an isomorphism) an extension of R.

Let $F(R)$ be the set of formulas in $L(R)$ with one free variable x. For ϕ, ψ, in $F(R)$ we set $\phi \sim \psi$, if and only if $K' \cup D(R) \vdash \phi \leftrightarrow \psi$. This gives us an equivalence relation on $F(R)$; we denote the equivalence class of ϕ by $[\phi]$. The equivalence classes form a Boolean algebra $B(R)$ (called the Lindenbaum algebra of $K' \cup D(R)$) under the operations $[\phi] + [\psi] = [\phi \vee \psi]$, $[\phi]' = [\neg \phi]$. We call the elements of the Stone space $S(R)$ of $B(R)$ 1-types over R. Notice that a point $p \in S(R)$ is isolated in the Stone topology if and only if there is a formula ϕ such that $K' \cup D(R) \vdash \phi \to \psi$ for every ψ such that $[\psi] \in p$, and $[\phi] \neq 0$ in $B(R)$. Such a formula ϕ is called a generator for p.

Since $K' \cup D(R)$ is complete, it is clear that if $p \in S(R)$ is isolated, then every model A of $K' \cup D(R)$ realizes p, i.e. there exists $a \in A$ such that $\{[\phi]: A \models \phi(a)\} = p$.

We shall present our results within the framework of finite forcing relative to $K \cup D(R)$ (for R countable). The fundamental papers on finite forcing in model theory are [1] and [4]. We expand $L(R)$ to $L(R, C)$ by adding a countable set C of new constant symbols. In our setting, a forcing condition $q(\underline{a}_1, \cdots, \underline{a}_m, \underline{c}_1, \cdots, \underline{c}_n)$ is a finite set of polynomial equations and inequations in the language $L(R, C)$ which is consistent with $K \cup D(R)$, i.e. such that there exists a model A of K extending R and $c_1, \cdots, c_n$ in A such that A satisfies all the statements in q when each $\underline{a}_i$ is interpreted as a_i, and each $\underline{c}_i$ is interpreted as c_i. If q is a condition and ϕ is a sentence in $L(R, C)$, then the notion "q forces ϕ" is defined by induction on the structure of ϕ, as follows:

(i) If ϕ is an equation, q forces ϕ if and only if $\phi \in q$;

(ii) q forces $\phi \wedge \psi$ if and only if q forces ϕ and q forces ψ;

(iii) q forces $\phi \vee \psi$ if and only if q forces ϕ or q forces ψ or both;

(iv) q forces $\exists x \phi(x)$ if and only if for some $\underline{c} \in C$, q forces $\phi(\underline{c})$;

(v) q forces $\neg\phi$ if and only if for no condition q' extending q is it the case that q' forces ϕ.

A sequence $\{q_i\}_{i=1}^{\infty}$ of conditions is *complete* if for each sentence ϕ of $L(R, C)$ there is some q_i which forces either ϕ or $\neg\phi$. A complete sequence determines in a canonical way (see [1] or [4]) a ring A which contains R. Every element of A is named by some $\underline{c}$ in such a way that all the statements in any q_i hold in A (when $\underline{a}$ is interpreted as a for $a \in R$), and any equation which holds in A is in some q_i. A is called finitely generic for $K \cup D(R)$. Since $K \cup D(R)$ has a model completion (namely $K' \cup D(R)$), this implies [1] that A is a model of $K' \cup D(R)$; in particular A is a model of K'.

We can now prove

Theorem 1. *There exists a countable model R of K which has no prime extension to a model of K'. Moreover, R can be chosen so that all the isolated points in $S(R)$ are realized in R, i.e. have a generator $x = \underline{r}$ for some $r \in R$. In particular the isolated points are not dense in $S(R)$.*

Corollary. *K' is not quasi-totally transcendental. (For this notion see* [5].)

Remark. It is easy to see that K' is κ-unstable for all infinite cardinals κ.

Proof of Theorem 1. Let R' be the ring of all locally constant functions from the Cantor space X into $\overline{Q}$, an algebraic closure of the rationals Q. (A function f on X is *locally constant* if for every $x \in X$ there exists a neighborhood U of x on which f is constant.) R' is a model of K' and R' is countable. For, notice that any locally constant function f on X determines a partition of X into finitely many clopen sets P_i such that f is constant on each P_i. Since $\overline{Q}$ is countable, any such partition is determined in this way by only countably many f's; and there are only countably many such partitions of X.

Pick a point $x_0 \in X$. Let $R \subset R'$ consist of all the elements $f \in R'$ such that $f(x_0) \in Q$. It is easy to see that R is a regular ring.

Now suppose R has a prime extension M to a model of K'. We can assume $R \subseteq M \subseteq R'$, and then we know that M is an elementary substructure of R'. Since R is not a model of K', we can pick $d \in M - R$.

Let p be the point of $S(R)$ realized by d in M. We will show that there exists a model of $K' \cup D(R)$, no element of which realizes p; it follows that M cannot be embedded in this model over R, so M is not prime.

Let C, as before, be a countable set of new constants not in $L(R)$, and let $\{\phi_i\}_{i=1}^{\infty}$ be an enumeration of the sentences in $L(R, C)$. We will in a moment define a complete sequence $\{q_i\}$ of forcing conditions, but first we state the following

Lemma. *Let q be a condition. Let $\underline{c}_1, \cdots, \underline{c}_n$ be the elements of C mentioned in q. Then there exist elements $a_1, b_1, \cdots, a_n, b_n$ in R such that $q \cup \{\underline{b}_i \neq \underline{c}_i \underline{a}_i \mid i = 1, \cdots, n\}$ is a condition, where $a_i, b_i \in R$ are such that $b_i = da_i$ in R'.*

We will prove the Lemma later.

Now define a sequence $\{q_i\}$ of conditions as follows: Let q_1 be a condition which forces either ϕ_1 or $\neg\phi_1$. Let q_2 be an extension of q_1 obtained by using the Lemma. If $i > 1$ is odd, let q_i be an extension of q_{i-1} which forces either $\phi_{(i+1)/2}$ or $\neg\phi_{(i+1)/2}$. If $i > 1$ is even, let q_i be an extension of q_{i-1} obtained by using the Lemma. As indicated above, the complete sequence $\{q_i\}$ determines a model R'' of K' which extends R, and every element of R'' is denoted by some $\underline{c} \in C$.

Suppose some element $r \in R''$ realizes p, and that this element is denoted by $\underline{c}_i$. Then for some odd j, $\underline{c}_i$ appears in q_j. Thus there exist a_i, b_i in R such that $b_i = da_i$ in R' and q_{j+1} contains the statement $\underline{b}_i \neq \underline{c}_i \underline{a}_i$. Since $\underline{b}_i \neq \underline{c}_i \underline{a}_i$ is in q_{j+1}, $\underline{b}_i \neq x\underline{a}_i$ is in the type realized by r in R''. But $\underline{b}_i = x\underline{a}_i$ is in p, since p is the type realized by d in R' and $b_i = da_i$. Thus r does not realize p in R''. This proves the first statement of the theorem.

To prove the second statement, suppose there is an isolated point p in $S(R)$ which is not realized in R. Since p is isolated there is d in $R' - R$ which realizes p. Then by the above, there exists a model of $K' \cup D(R)$, no element of which realizes p. Hence p is not isolated, a contradiction.

To see that the isolated points are not dense in $S(R)$, observe that if $\phi(x)$ is a formula in $L(R)$ such that $K' \cup D(R) \vdash \exists x\phi(x)$, but $R \models \neg \exists x\phi(x)$, then the neighborhood in $S(R)$ determined by $\phi(x)$ contains no isolated points, since all the isolated points are realized in R. An example of such a formula is $x^2 = 2$.

This finishes the proof, modulo the Lemma.

In proving the Lemma we will work with idempotents of R', and it will be helpful to make some preparatory remarks about them. An idempotent e of R' is an element e in R' whose values are everywhere either 0 or 1. We identify e with the clopen subset of X consisting of all points $x \in X$ such that $e(x) = 1$. This gives a 1-1 correspondence between the idempotents of R' and the clopen subsets of X. Thus if e, f are idempotents, we say $e \subset f$ if $e(x) = 1$ implies $f(x) = 1$ for all $x \in X$. Furthermore the correspondence makes it clear what we mean when we say that an element of R' is constant on some idempotent.

Let q be a condition, let $h_1, \cdots, h_m$ be all the elements of R such that $\underline{h}_i$ occurs in q, and let $\underline{c}_1, \cdots, \underline{c}_n$ be the elements of C which occur in q. Let $\phi(x_1, \cdots, x_n)$ be the formula obtained by replacing $\underline{c}_i$ by x_i in the conjunction of the elements of q. Then since q is a condition, the formula $\exists x_1 \cdots \exists x_n \phi$ holds in some model of $K' \cup D(R)$, and hence in all of them, since K' is the model completion of K. In particular $R' \models \exists x_1 \cdots \exists x_n \phi$, so there exist elements $c_1, \cdots, c_n$ in R' such that $R' \models \phi(\underline{c}_1, \cdots, \underline{c}_n)$ when $\underline{c}_i$ is interpreted as c_i and $\underline{h}_i$ is interpreted as h_i.

$\phi(\underline{c}_1, \cdots, \underline{c}_n)$ is a conjunction of polynomial equations $\{P_{i,1} = P_{i,2}\}_{i=1}^{s}$ and inequations $\{P_{s+i,1} \neq P_{s+i,2}\}_{i=1}^{t}$ which hold in R'. For each i, $1 \leq i \leq t$, let $x_i \neq x_0$ be a point in X at which $P_{s+i,1} \neq P_{s+i,2}$. Let e be an idempotent of R such that $e(x_0) = 1$ and $e(x_i) = 0$ for $1 \leq i \leq t$. (The fact that e exists follows from the properties of X.) Let $f \subset e$ be an idempotent such that $f(x_0) = 1$ and each of $d, h_1, \cdots, h_m, c_1, \cdots, c_n$ is constant on f. Notice that since each $h_i \in R$, each h_i has a constant *rational* value on f.

Let d^* denote the constant irrational value of d on f. Let u, v in R be idempotents such that $u, v \subset f$, $u(x_0) = v(x_0) = 0$, $uv = 0$, and $u \neq 0, v \neq 0$. Let y, z in R be such that $y(x) = d^*$ for all $x \in u$ and $y(x) = 0$ for all $x \notin u$, $z(x) = d^*$ for all $x \in v$, $z(x) = 0$ for all $x \notin v$.

Since d^* is irrational, there exists an automorphism ρ of $\overline{Q}$ such that $\rho(d^*) \neq d^*$. Denote the constant values of the h_i and c_i on f by h_i^*, c_i^*, respectively. Then each $h_i^* \in Q$, so $\rho(h_i^*) = h_i^*$ for each i. Let c_i', $1 \leq i \leq n$, be the element of R' which has the constant value $\rho(c_i^*)$ on u and the same values as c_i on $1 - u$.

Let $J = \{j \in \{1, \cdots, n\} | c_j$ has the constant value d^* on $f\}$. For $i \in J$ let $a_i = u$ and let $b_i = y$. For $i \notin J$ let $a_i = v$ and let $b_i = z$.

We claim that $q \cup \{\underline{b}_i \neq \underline{c}_i\,\underline{a}_i \mid 1 \leq i \leq n\}$ holds in R' when we interpret $\underline{a}_i$ as a_i, $\underline{b}_i$ as b_i, $\underline{h}_i$ as h_i, and $\underline{c}_i$ as c_i'. The fact that the part in brackets holds follows immediately from the definition of the a_i, b_i, and c_i'. The inequalities in q hold, because the c_i' agree with the c_i on $x_1, \cdots, x_t$. The equalities hold, because the c_i' differ from the c_i only on u, and on u the equalities hold for the c_i', because ρ is an automorphism of $\overline{Q}$.

This completes the proof.

Remarks. (1) Using the model completeness of K', one can argue directly from the Lemma that the type of d is not principal; one can then fall back on the standard omitting types theorem to conclude that R has no prime extension to a model of K'. However we feel that the presentation in terms of forcing is more intuitive and more nearly self-contained.

(2) In the general framework of [5], one considers theories which are model completions of universal theories.

In any commutative regular ring there exists for any element x a unique element $f(x)$ such that $x^2 f(x) = x$ and $f(x)^2 x = f(x)$ (see [2]). If we enlarge our language L to L' by adding a 1-place function symbol f, and write the axiom of regularity in the form

$$\forall x (x^2 f(x) = x \wedge f(x)^2 x = f(x)),$$

then, as is remarked in [3], K' is in L' the model completion of K, and K in L' is a universal theory. Since the function f is definable in L, it is easy to see that Theorem 1 holds for L' as well as for L. Thus with respect to L', K' is a natural example of a model completion of a universal theory K which has a countable model R such that the isolated points are not dense in $S(R)$.

Recall that K^* is K' with axiom (ii) deleted.

Theorem 2. *If R is the ring of Theorem* 1, *then R has no prime extension to a model of K^*.*

Proof. Suppose A is a prime extension of R to a model of K^*. Then since R' (as defined in the proof of Theorem 1) is a model of K^*, there exists an embedding $f: A \to R'$ such that $f|R$ is the identity map. Therefore A has the same idempotents as R, since R' has the same idempotents as R. Thus there are no minimal idempotents in A, so A is in fact a model of K'. Thus A is a prime extension of R to a model of K', contradicting Theorem 1.

We should point out that R' is a minimal extension of R to a model of

K' (and K^*). That is, there is no model of K' (or K^*) sitting strictly between R and R'. For suppose S is a model of K^* and $R \subseteq S \subseteq R'$. Then for any $x \in X$, $\{f(x) | f \in S\} = \bar{Q}$. Now let $f \in R'$; we claim $f \in S$. To see this, we observe that if $x \in X$ there exists an idempotent e_x containing x and $f_x \in S$ such that f_x and f have the same constant value on e_x. It follows from the properties of X that there exist disjoint idempotents $e_1, \cdots, e_n$ in R and elements $f_1, \cdots, f_n$ in S such that $\Sigma_{i=1}^n e_i = 1$, and $f_i e_i = fe_i$, $1 \leq i \leq n$. Then

$$f = \sum_{i=1}^{n} fe_i = \sum_{i=1}^{n} f_i e_i \in S.$$

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DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520

Current address (D. Saracino): Department of Mathematics, Colgate University, Hamilton, New York 13346

Current address (V. Weispfenning): Mathematisches Institut der Universität, Heidelberg, West Germany