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GALOIS THEORY AND THE EXISTENCE OF MAXIMAL UNRAMIFIED SUBALGEBRAS[1]

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ABSTRACT. Let B be a commutative ring with 1, let G be a finite group of automorphisms of B, and let A be the subring of G-invariant elements of B. There exists a G-stable, unramified A-subalgebra of B which contains every unramified A-subalgebra of B.

Throughout this paper B will denote a given commutative ring with 1. G will denote a given finite group of automorphisms of B, and A will denote the subring of G-invariant elements of B. Following the terminology of [1], an A-subalgebra A' of B will be called unramified if $A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}}$ is a separable algebra over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ in A, where $A_{\mathfrak{p}}$ (resp. $A'_{\mathfrak{p}}$) is the ring of fractions of A (resp. A') with respect to the complement of $\mathfrak{p}$ in A.

Lemma. *Let $\mathfrak{m}$ be a maximal ideal of A, and suppose A' is an A-subalgebra of B such that $A'/A'\mathfrak{m}$ is a separable $A/\mathfrak{m}$-algebra.*

(i) *The homomorphism of $A'/A'\mathfrak{m}$ into $B/B\mathfrak{m}$ induced by the inclusion map of A' into B is an injection, by which $A'/A'\mathfrak{m}$ may be identified with a subalgebra of $B/B\mathfrak{m}$.*

(ii) *The dimension of the algebra $A'/A'\mathfrak{m}$ over the field $A/\mathfrak{m}$ does not exceed the order of G.*

(iii) *$A'/A'\mathfrak{m}$ and the subring of G-invariant elements of $B/B\mathfrak{m}$ are linearly disjoint subalgebras of the $A/\mathfrak{m}$-algebra $B/B\mathfrak{m}$.*

Proof. Note that $A'/A'\mathfrak{m}$ is a finite-dimensional algebra over the field $A/\mathfrak{m}$. More generally, a separable algebra over a commutative ring which is a projective module over that ring is finitely generated as a module by [6, Proposition 1.1]. Therefore $A'/A'\mathfrak{m}$ is a semisimple algebra by [4, Chapter

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IX, Proposition 7.7 and Theorem 7.10], and $A'\mathfrak{m}$ must equal the intersection of the maximal ideals of A' which contain it. Since B is integral over A [3, Chapter V, §1, Proposition 22], B is integral over A'. Since every prime ideal of A' is the contraction of a prime ideal of B [3, Chapter V, §2, Theorem 1], it follows that $A'\mathfrak{m}$ is the contraction of some ideal $\mathfrak{R}$ of B, $B\mathfrak{m} \subseteq \mathfrak{R}$, and $A' \cap B\mathfrak{m} \subseteq A' \cap \mathfrak{R} = A'\mathfrak{m}$. But obviously $A'\mathfrak{m} \subseteq A' \cap B\mathfrak{m}$ and, therefore, $A'\mathfrak{m} = A' \cap B\mathfrak{m}$ and the homomorphism of $A'/A'\mathfrak{m}$ into $B/B\mathfrak{m}$ induced by the inclusion map of A' into B is injective.

Let $B' = \Pi_{\sigma \in G}\sigma(A')$, and let H be the group of automorphisms of B' which are restrictions of elements of G. Since each element σ of G induces an $A/\mathfrak{m}$-algebra isomorphism of $A'/A'\mathfrak{m}$ onto $\sigma(A')/\sigma(A')\mathfrak{m}$, $\sigma(A')/\sigma(A')\mathfrak{m}$ is again a separable $A/\mathfrak{m}$-algebra, and $B'/B'\mathfrak{m}$, which is a homomorphic image of the tensor product of the $A/\mathfrak{m}$-algebras $\sigma(A')/\sigma(A')\mathfrak{m}$, $\sigma \in G$, is a separable algebra over $A/\mathfrak{m}$ by [2, Propositions 1.4 and 1.5]. Consequently, $B'\mathfrak{m}$ must equal the intersection of the maximal ideals of B' which contain it. Because $\mathfrak{m}$ is a maximal ideal of A, the set of maximal ideals of B' which contain $B'\mathfrak{m}$ coincides with the set of maximal ideals of B' which lie over $\mathfrak{m}$. Choose a maximal ideal $\mathfrak{M}_0$ of B' which lies over $\mathfrak{m}$, let $H^Z(\mathfrak{M}_0)$ be the subgroup of $\sigma \in H$ such that $\sigma(\mathfrak{M}_0) \subseteq \mathfrak{M}_0$, and let $H^T(\mathfrak{M}_0)$ be the subgroup of $\sigma \in H^Z(\mathfrak{M}_0)$ which induces the identity automorphism on $B'/\mathfrak{M}_0$. By [3, Chapter V, §2, Theorem 2], H acts transitively on the set of all prime ideals of B' which lie over $\mathfrak{m}$, and $B'/\mathfrak{M}_0$ is a normal field extension of $A/\mathfrak{m}$ with Galois group isomorphic to the quotient group $H^Z(\mathfrak{M}_0)/H^T(\mathfrak{M}_0)$. Therefore the prime ideals of B' which lie over $\mathfrak{m}$ are maximal, their number is finite and equal to $(H\colon H^Z(\mathfrak{M}_0))$, and $B'/\mathfrak{M}$ is isomorphic to $B'/\mathfrak{M}_0$ for every maximal ideal $\mathfrak{M}$ of B' which lies over $\mathfrak{m}$. $B'/\mathfrak{M}_0$ is a separable field extension of $A/\mathfrak{m}$ by [2, Proposition 1.4], and so the dimension of $B'/\mathfrak{M}_0$ over $A/\mathfrak{m}$ is equal to the order of the Galois group of $B'/\mathfrak{M}_0$ over $A/\mathfrak{m}$. Letting $\mathfrak{M}$ range over the set of maximal ideals of B' which contract to $\mathfrak{m}$, $B'/B'\mathfrak{m}$ is isomorphic to the direct product of the fields $B'/\mathfrak{M}$ [3, Chapter II, §1, Proposition 5], and the dimension of $B'/B'\mathfrak{m}$ over $A/\mathfrak{m}$ must equal

$$[H : H^Z(\mathfrak{M}_0)] \cdot [H^Z(\mathfrak{M}_0) : H^T(\mathfrak{M}_0)] = [H : H^T(\mathfrak{M}_0)].$$

Use the homomorphisms induced by the inclusion maps of A' into B' and B' into B to identify $B'/B'\mathfrak{m}$ with a subalgebra of $B/B\mathfrak{m}$ and $A'/A'\mathfrak{m}$ with a subalgebra of $B'/B'\mathfrak{m}$. Then neither the dimension of the $A/\mathfrak{m}$-algebra $B'/B'\mathfrak{m}$ nor the dimension of its subalgebra $A'/A'\mathfrak{m}$ can exceed the order of G.

Finally, letting $\bar{A}$ be the subring of G-invariant elements of $B/B\mathfrak{m}$, it is evident that $\bar{A}$ is an $A/\mathfrak{m}$-algebra. If the canonical homomorphism of $(B'/B'\mathfrak{m})\otimes_{A/\mathfrak{m}}\bar{A}$ into $B/B\mathfrak{m}$, which maps $b\otimes a$ onto ba for $b\in B'/B'\mathfrak{m}$ and $a\in\bar{A}$, is injective, then $B'/B'\mathfrak{m}$ and $\bar{A}$ are linearly disjoint subalgebras of the $A/\mathfrak{m}$-algebra $B/B\mathfrak{m}$, and, consequently, so are $A'/A'\mathfrak{m}$ and A. But $B/B\mathfrak{m}\approx(B'/B'\mathfrak{m})\otimes_{B'}B$, and it has been noted already that $B'/B'\mathfrak{m}$ is a direct product of the fields $B'/\mathfrak{M}$, $\mathfrak{M}$ ranging over the set of maximal ideals of B' which contract to $\mathfrak{m}$. Therefore, letting $\mathfrak{M}_0$ be any given maximal ideal of B' which lies over $\mathfrak{m}$, it is sufficient to prove that the canonical homomorphism π of $(B'/\mathfrak{M}_0)\otimes_{A/\mathfrak{m}}\bar{A}$ into $B/B\mathfrak{M}_0\approx(B'/\mathfrak{M}_0)\otimes_{B'}B$, which maps $b\otimes a$ onto ba for $b\in B'/\mathfrak{M}_0$ and $a\in\bar{A}$, is injective. Since $B'/\mathfrak{M}_0$ is a normal, separable field extension of $A/\mathfrak{m}$ with Galois group $H^Z(\mathfrak{M}_0)/H^T(\mathfrak{M}_0)$, there exist a positive integer n and elements x_i, y_i of $B'/\mathfrak{M}_0$, $1\le i\le n$, such that $\sum_{i=1}^n x_i\cdot\rho(y_i)=\delta_{1,\rho}$ for all $\rho\in H^Z(\mathfrak{M}_0)/H^T(\mathfrak{M}_0)$ by [5, Theorem 1.3]. Letting $\tau\in H^Z(\mathfrak{M}_0)$ and letting σ be an element of G which extends τ, σ induces an $\bar{A}$-algebra automorphism on the image of π, and in this way $H^Z(\mathfrak{M}_0)$ is represented as a group of automorphisms of the image of π. Moreover, $H^T(\mathfrak{M}_0)$ is the kernel of this representation, and thus $H^Z(\mathfrak{M}_0)/H^T(\mathfrak{M}_0)$ is represented as a group of $\bar{A}$-algebra automorphisms of the image of π. For any element c of the image of π, let $\operatorname{tr}(c)$ be the sum of the elements $\rho(c)$, $\rho\in H^Z(\mathfrak{M}_0)/H^T(\mathfrak{M}_0)$, and notice that, if $c\in B'/\mathfrak{M}_0$, then $\operatorname{tr}(c)\in A/\mathfrak{m}$. If $b\in B'/\mathfrak{M}_0$ and $a\in\bar{A}$, then

$$b\otimes a=\sum_{i=1}^n x_i\cdot\operatorname{tr}(y_ib)\otimes a=\sum_{i=1}^n x_i\otimes\operatorname{tr}(y_iba)\quad\text{in } (B'/\mathfrak{M}_0)\otimes_{A/\mathfrak{m}}\bar{A};$$

and from this formula it follows easily that π is injective.

Theorem. *There exists an unramified A-subalgebra of B which is stable under G and contains every unramified A-subalgebra of B.*

Proof. Let $\mathfrak{p}$ be any prime ideal of A, and let A' be an unramified A-subalgebra of B. Then $A_\mathfrak{p}$ is the subring of G-invariant elements of $B_\mathfrak{p}$ by [3, Chapter V, §1, Proposition 23], $\mathfrak{p}A_\mathfrak{p}$ is a maximal ideal of $A_\mathfrak{p}$, and $A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p}$ is a separable $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-algebra. Therefore, the inclusion map of $A'_\mathfrak{p}$ into $B_\mathfrak{p}$ induces a monomorphism by which $A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p}$ may be identified with a subalgebra of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ and the dimension of $A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p}$ over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ does not exceed the order of G by the preceding lemma. Partially order the unramified A-subalgebras of B by inclusion, let $\mathfrak{F}$ be a chain of unramified A-subalgebras of B, and let $\bar{A}=\bigcup_{A'\in\mathfrak{F}}A'$. Choose an element A' of $\mathfrak{F}$ for which the dimension of $A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p}$ over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$

is greatest. If B' is an element of $\mathfrak{F}$ such that $A' \subseteq B'$, then the dimensions of the $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-algebras $A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}}$ and $B'_{\mathfrak{p}}/\mathfrak{p}B'_{\mathfrak{p}}$ must be equal, and therefore $A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}} = B'_{\mathfrak{p}}/\mathfrak{p}B'_{\mathfrak{p}}$. Consequently, $\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}} = A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}}$, and so $\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}}$ is a separable $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-algebra. Thus $\bar{A}$ is an unramified A-subalgebra of B, and certainly it is an upper bound for $\mathfrak{F}$. By Zorn's lemma, there exists a maximal unramified A-subalgebra C of B. If A' is any unramified A-subalgebra of B, then $(A'C)_{\mathfrak{p}}/\mathfrak{p}(A'C)_{\mathfrak{p}}$, which is a homomorphic image of the tensor product of the $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-algebras $A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}}$ and $C_{\mathfrak{p}}/\mathfrak{p}C_{\mathfrak{p}}$, is a separable algebra over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A, and consequently $A'C$ is an unramified A-subalgebra of B which contains C. Therefore, $A'C = C$ and $A' \subseteq C$. If $\sigma \in G$, then $\sigma(C)$ is again an unramified A-algebra, and so $\sigma(C) \subseteq C$. Therefore, C is stable under G.

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