HARRISON'S WITT RING OF A GENERALIZED VALUATION RING

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ABSTRACT. The purpose of this paper is to determine the relations between $H(R)$ (Harrison's Witt ring given by generators and relations over a generalized valuation ring R) and $W(K)$ (classical Witt ring of inner product spaces over the field K). Using the main results, we are able to establish some theorems on the structure of $H(R)$ analogous to the field case.

1. Preliminaries. For a commutative ring R , Harrison's Witt ring, $H(R)$, is the ring given by generators $\langle a \rangle$, $a \in R$, and relations

$$(H1) \langle 0 \rangle = 0,$$

$$(H2) \langle ab \rangle = \langle a \rangle \langle b \rangle, \text{ and}$$

$$(H3) \langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle (a + b)ab \rangle.$$

It is shown in [3] and [4] that for a field F , $H(F)$ is the classical Witt ring of F , $W(F)$, based on inner product spaces over F .

An H -map is a mapping $t: R \rightarrow S$ of commutative rings satisfying $t(0) = 0$, $t(ab) = t(a)t(b)$ and $t(a) + t(b) = t(a + b) + t((a + b)ab)$. Thus $\langle \rangle_R$ (or $\langle \rangle$): $R \rightarrow H(R)$ is a universal H -map.

H is a functor from the category of commutative rings to the same category, where for a ring homomorphism $f: R \rightarrow S$, $H(f): H(R) \rightarrow H(S)$ is the unique ring homomorphism such that $\langle \rangle_S \circ f = H(f) \circ \langle \rangle_R$.

1.1 Lemma. *An element r in R is a unit if and only if $\langle r^2 \rangle = 1$ in $H(R)$.*

Proof. Apply Lemmas 2.1, 2.3 and 3.1 in [1]. $\square$

Let L_R (or L) denote the ideal of R generated by $x^2y + xy^2$ for all x, y in R .

1.2 Lemma. *If L contains a unit, then $\langle r \rangle = \langle r^3 \rangle$ for all r in R .*

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Proof. Apply Lemmas 3.1 and 3.4 in [1]. $\square$

2. Jordan decomposition of an element in $H(R)$. From now on we assume that R is a generalized valuation ring, i.e. R is a commutative, noetherian ring with identity whose ideals are totally ordered with respect to inclusion. Choose a prime π in R and fix it once and for all. Every ideal of R is of the type $\pi^n R$ for some nonnegative integer n . Let k_R (or k) denote the residue class field of R (i.e. $k \simeq R/\pi R$). If R is an integral domain, we denote the quotient field of R by F_R (or F). Let $- : R \rightarrow F$ be the canonical maps. We use $-$ and i to denote the induced maps $H(-)$ and $H(i)$, respectively.

2.1 Theorem. *Suppose R is not an integral domain. Then*

$$H(R) \simeq \begin{cases} Z_2 \times Z_2 & \text{if } k = Z_2, \\ W(k) & \text{if } k \neq Z_2. \end{cases}$$

Proof. Suppose $k = Z_2$. Let $Z_2(C_2)$ denote the group ring of the group $C_2 = \{1, x\}$ over Z_2 . Define the map $t: R \rightarrow Z_2(C_2)$ as follows:

$$t(r) = \begin{cases} 1 & \text{if } r \in R, \\ x & \text{if } r \in \pi R \setminus \pi^2 R, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that t is an H -map, and the homomorphism induced by t is an isomorphism, so $H(R) \simeq Z_2(C_2)$.

Suppose $k \neq Z_2$. The radical of R equals πR . Since L contains a unit, by 1.2 and [1, 2.1], $\langle \pi \rangle = 0$. It follows from [1, 3.1] that $W(k) \simeq H(R/\pi R) \simeq H(R)/\langle \pi \rangle H(R) \simeq H(R)$. $\square$

Theorem 2.1 determines $H(R)$ completely when R is not an integral domain. So for the remainder of this paper we assume that R is an integral domain.

Let $\dot{R}$ be the multiplicative group of units of R . We denote $\langle a_1 \rangle + \cdots + \langle a_n \rangle$ by $\langle a_1, \dots, a_n \rangle$. An element V in $H(R)$ is called unimodular provided $V = 0$, or there exists $a_1, \dots, a_n \in \dot{R}$ such that $V = \langle a_1, \dots, a_n \rangle$, and if $b_1, \dots, b_n \in R$ with $\sum a_i b_i^2 \in \pi R$, then $b_i \in \pi R$ for all $i = 1, \dots, n$. Since

$$(a_1 + a_2)b_1^2 + (a_1 + a_2)a_1 a_2 b_2^2 = a_1(b_1 + a_2 b_2)^2 + a_2(b_1 - a_1 b_2)^2,$$

$\langle a_1, a_2 \rangle$ is unimodular if and only if $\langle 1, a_1 a_2 \rangle \langle a_1 + a_2 \rangle$ is unimodular.

Hence unimodularity is well defined.

Let I_R (or I) be the ideal of $H(R)$ generated by $\langle 1, -1 + a\pi \rangle$ for all $a \in R$. Let j_r (or j): $I_R \rightarrow H(R)$ be the canonical embedding.

2.2 Lemma. $\langle \pi \rangle = \langle \pi^3 \rangle$ if and only if $k \neq Z_2$.

Proof. Sufficiency follows from 1.2. Conversely if $k = Z_2$ then $H(R)/\langle \pi \rangle H(R) \simeq H(k) \simeq W(k) \simeq Z_2$. On the other hand $H(R)/\langle \pi^3 \rangle H(R) \simeq H(R/\pi^3 R) \simeq Z_2 \times Z_2$ by 2.1. Hence $\langle \pi \rangle \neq \langle \pi^3 \rangle$. $\square$

2.3 Lemma. Suppose V is a unimodular element in $H(R)$. If $V \in I$ then $V = 0$.

Proof. It follows directly from definitions. $\square$

We let $s = 2$ if $k \neq Z_2$ and $s = 3$ if $k = Z_2$.

2.4 Proposition. Let $V \in H(R)$. Then there exist unimodular elements $V_0, V_1, \dots, V_s \in H(R)$ and $V_{s+1} \in I$ such that

$$(2.1) \quad V = V_0 + \langle \pi \rangle V_1 + \dots + \langle \pi^s \rangle V_s + V_{s+1}.$$

Proof. Let $V = \langle a_1, \dots, a_n \rangle$ with $a_i \in R$ for $i = 1, \dots, n$. We use induction on n . If $n = 1$ it follows from 1.2 and [1, 2.1]. We assume $n > 1$. Let $\langle a_1, \dots, a_{n-1} \rangle = U_0 + \dots + \langle \pi^s \rangle U_s + U_{s+1}$, where $U_0, \dots, U_s$ are unimodular elements and $U_{s+1} \in I$. By 1.2 and [1, 2.1], a_n can be chosen to be $\pi^i b$, where $b \in R$ and $0 \leq i \leq s$. Let $U_i = \langle b_1, \dots, b_m \rangle$ where $b_1, \dots, b_m$ are units. If $U_i + \langle b \rangle$ is unimodular, then it is done. If $U_i + \langle b \rangle$ is not unimodular there exist units $c_{j_1}, \dots, c_{j_l}$, with $1 \leq l \leq m$, such that $b_{j_1} c_{j_1}^2 + \dots + b_{j_l} c_{j_l}^2 \equiv -b \pmod{\pi R}$. By using $b_{j_\alpha} c_{j_\alpha}^2$ instead of b_{j_α} , $\alpha = 1, \dots, l$, and by rearranging the b_{j_α} 's, we may assume, without loss of generality, that $b_1 + \dots + b_l \equiv -b \pmod{\pi R}$. Hence $U_i + \langle b \rangle = U'_i + \langle b_1 + \dots + b_l, b \rangle$, where

$$U'_i = \sum_{j=2}^l \langle (b_1 + \dots + b_{j-1}) b_j (b_1 + \dots + b_j) \rangle + \langle b_{l+1}, \dots, b_m \rangle.$$

We only have to show that U'_i is unimodular. Let $b'_j = (b_1 + \dots + b_{j-1}) \cdot b_j (b_1 + \dots + b_j)$ for $j = 2, \dots, l$, and $b'_j = b_j$ for $j = l+1, \dots, m$. Let $d = \sum_{j=2}^m d_j^2 b'_j$ where $d_j \in R$. It follows from induction on l and straightforward computations that

$$d = b_1 \left(\sum_{j=2}^l d_j b_j \right)^2 + \sum_{j=2}^l b_j \left(-d_j(b_1 + \cdots + b_{j-1}) + \sum_{\alpha=j+1}^l b_\alpha d_\alpha \right)^2 + \sum_{j=l+1}^m b_j d_j^2.$$

If $d \in \pi R$, then by the unimodularity of U_i ,

$$\sum_{\alpha=j+1}^l b_\alpha d_\alpha - d_j(b_1 + \cdots + b_{j-1}) \in \pi R$$

for $j = 2, \dots, l$, and $d_j \in \pi R$ for $j = l+1, \dots, m$. Solving the system of $(l-1)$ linear equations in $(l-1)$ unknowns, $d_2, \dots, d_l$, it follows that $d_i \in \pi R$ for $i = 2, \dots, l$. Hence U_i is unimodular. $\square$

For each $V \in H(R)$, the decomposition (2.1) of V is called a Jordan decomposition of V . The decomposition depends on the choices of the prime element π in R , and it is not even unique with respect to a fixed prime π . However we will show that the decomposition is unique modulo I .

2.5 Lemma. *Let $V = \sum_{m=1}^s \langle \pi^m \rangle V_m \in H(R)$, where V_m is unimodular for $m = 1, \dots, s$. Then $V \in I$ if and only if $V_m = 0$ for $m = 1, \dots, s$.*

Proof. Necessity follows from the unimodularity of V_m and 2.3. Sufficiency is obvious. $\square$

2.6 Theorem. *Let $V \in H(R)$ and let*

$$V = \sum_{m=0}^s \langle \pi^m \rangle V_m + V_{s+1} = \sum_{m=0}^s \langle \pi^m \rangle U_m + U_{s+1}$$

be two Jordan decompositions of V . Then $\bar{V}_m = \bar{U}_m$ in $W(k)$ for $m = 0, \dots, s$.

Proof. It is easy to see that $\bar{V}_0 = \bar{V} = \bar{U}_0$ in $H(k) = W(k)$. Thus $V_0 - U_0 \in I$ and $\sum_{m=1}^s \langle \pi^m \rangle (V_m - U_m) \in I$. Since $V_m - U_m$ is of the form $\sum \langle u_j \rangle$, u_j units, it can be written as $W_0 + W_1$, where W_0 is unimodular and $W_1 \in I$. Then apply 2.5. $\square$

3. Associate $H(R)$ with $W(k)$ and $W(F)$. Let $V \in H(R)$ with Jordan decomposition (2.1). Define a map $\rho: H(R) \rightarrow \bigoplus_{m=0}^s W(k)$ by letting $\rho(V) = (\bar{V}_0, \dots, \bar{V}_s)$. By 2.5, ρ is well defined.

Let $V = \sum \langle a_m \rangle \langle 1, -1 + b_m \rangle \in I$. Define a map $\eta: I \rightarrow W(F)$ by letting $\eta(V) = \sum \langle a_m \rangle_F \langle 1, -1 + v_m \rangle_F \in W(F)$.

In this section, we will show that $\ker \rho = I$ and η is injective. By using these results, we prove that $H_t(R)$, the torsion subgroup of $H(R)$, is 2-primary.

3.1 Proposition. ρ is a surjective additive homomorphism with kernel I .

Proof. It is easy to see that ρ is a surjective additive homomorphism and $\ker \rho \supseteq I$. $\ker \rho \subseteq I$ follows from 2.3 and 2.5. $\square$

3.2 Corollary. If R lifts squares (i.e. $1 + \alpha\pi$ is a square for $\alpha \in R$), then $I = \{0\}$ and $H(R) \simeq \bigoplus_{m=0}^s W(k)$.

3.3 Proposition. If D is a Dedekind domain with quotient field K , then $H(D) \rightarrow H(K) \times \pi H(D/P^2)$ is one-one, where P ranges over all nonzero prime ideals of D .

Proof. See [2, Corollary 1.6]. $\square$

3.4 Theorem. η is an injective ring homomorphism.

Proof. $\text{Ker}(H(R) \rightarrow H(R/\pi^2 R)) = \langle \pi^2 \rangle H(R) \supseteq \langle \pi^2 \rangle I_R = I_R$. By 3.3,

$$\text{Ker}(H(R) \rightarrow H(R/\pi^2 R)) \cap \text{Ker}(H(i)) = 0.$$

Thus $0 = I_R \cap \text{Ker } H(i) \supseteq I_R \cap \text{Ker } \eta$, so η is injective. $\square$

3.5 Theorem. The torsion subgroup of $H(R)$, $H_t(R)$, is a 2-primary subgroup.

Proof. Let K be a field. Then $W_t(K)$, the torsion subgroup of $W(K)$, is 2-primary. Let $V = \sum_{m=0}^s \langle \pi^m \rangle V_m + V_{s+1} \in H_t(R)$. Then each $\bar{V}_m \in W_t(k)$ for $m = 0, \dots, s$. By 3.1 and 3.4 there exists some nonnegative integer n with $2^n V \in I \cap H_t(R)$. By 3.2 and 3.4, $I \cap H_t(R)$ is 2-primary, so V has order a power of 2. $\square$

4. Transfer maps. Let R and S be two generalized valuation domains with R a subring of S . Let $f: R \rightarrow S$ be the canonical embedding. In this section, we try to find the kernel of $H(f)$ in certain special cases.

Thestufe of R , $s(R)$, is the smallest integer m such that $-1 = a_1^2 + \dots + a_m^2$ with $a_i \in R$ for $i = 1, \dots, m$. It has been shown in [5] that s is a power of 2.

A ring is called pythagorean if every sum of squares is a square itself. The pythagorean closure of a ring R , R_p , is the minimal pythagorean overring of R . If R is a generalized valuation domain, then R_p can be obtained by adjoining elements of the form $1 + a^2$ for $a \in R$.

4.1 Proposition. Suppose R, S are fields. If S/R is a finite extension of odd degree, then $H(f)$ is injective.

Proof. The same proof as that of [7, Theorem 2.2.2] works for arbitrary fields R and S . $\square$

4.2 Theorem. Suppose that S/R is a totally ramified extension when $k_R = Z_2$. If the rank of S over R is odd then $H(f)$ is injective.

Proof. It is easy to check that the following diagram commutes and the rows are exact by 3.1.

$$(4.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & I_R & \rightarrow & H(R) & \rightarrow & \bigoplus_{m=0}^s W(k_R) \rightarrow 0 \\ & & & & \downarrow W(f_F) & & \downarrow \bigoplus W(f_k) \\ 0 & \rightarrow & I_S & \rightarrow & H(S) & \rightarrow & \bigoplus_{m=0}^s W(k_S) \rightarrow 0 \end{array}$$

where $\bigoplus W(f_k)$ is well defined from the assumptions. By 4.1 and 3.4, $W(f_F)$ and $\bigoplus W(f_k)$ are injective. Hence $H(f)$ is injective. $\square$

4.3 Theorem. Suppose $k_R = Z_2$ and S/R is an extension of odd degree over R which is not totally ramified. Then $\text{Ker } H(f) = \langle \pi, -\pi^3 \rangle H(R)$.

Proof. It follows from the assumption that $k_S \neq Z_2$, thus $\langle \pi, -\pi^3 \rangle_S = 0$ by 1.2. This shows $\langle \pi, -\pi^3 \rangle H(R) \subseteq \text{Ker } H(f)$. Conversely, we define a map $g: \bigoplus_{m=0}^3 W(k_R) \rightarrow \bigoplus_{m=0}^2 W(k_S)$ by letting

$$g(\bar{V}_0, \dots, \bar{V}_3) = (\bar{V}_0, \bar{V}_1 + \bar{V}_3, \bar{V}_2) \in \bigoplus_{m=0}^2 W(k_S)$$

for every $(\bar{V}_0, \dots, \bar{V}_3) \in \bigoplus_{m=0}^3 W(k_R)$. By 4.1 and [1, 2.1, 2.3], g is a well-defined homomorphism with $\text{ker } g = \{(0, \bar{V}, 0, \bar{V}) \mid \bar{V} \in W(k_R)\}$. Use g instead of $\bigoplus W(f_k)$ in (4.1); then (4.1) still commutes since $\langle \pi \rangle = \langle \pi^3 \rangle$ in $H(S)$. By 4.1 and diagram chasing, $\langle \pi, -\pi^3 \rangle H(R) \supseteq \text{Ker } H(f)$. $\square$

4.4 Lemma. Let $S = R(\theta)$, where θ satisfies $x^2 - ax - 1 = 0$ for some $a \in R$. Then $\text{Ker}(H(f))$ is a torsion subgroup of $H(R)$.

Proof. Suppose R is a field. If $\text{char } R = 2$ then $H(R) = H_t(R)$; we are done. If $\text{char } R \neq 2$, then the lemma is Theorem 2.2.3 in [7].

In general, let $V \in \text{Ker}(H(f))$. If $\text{char } k_R = 2$, then $\rho_R(V) \in \bigoplus_{m=0}^s W_t(k_R)$ is obvious. If $\text{char } k_R \neq 2$, then it follows from 4.1 and the above discussion that $\rho_R(V) \in \bigoplus_{m=0}^2 W_t(k_R)$. Thus $n\rho_R(V) = 0 \in \bigoplus_{m=0}^s W(k_R)$. By the exactness of (4.1), $nV \in I_R$. $\eta_S \cdot w(f_F)(nV) = H(f)(nV) = 0$. So $W(f_F)(nV) = 0$ and nV is torsion. $\square$

4.5 Theorem. (a) If $s(R) < \infty$ then $H(R)$ is a torsion group.

(b) If $s(R) = \infty$ then $H_t(R) = \ker(H(f))$ where $f: R \rightarrow R_p$ is the canonical embedding.

Proof. If $s(R) < \infty$, then $s(F)$ and $s(k)$ are finite. Thus $W(F)$ and $W(k)$ are torsion. Apply 3.1 and 3.4.

Suppose $s(R) = \infty$. Let $V = \langle a_1, \dots, a_m \rangle \in H_t(V)$. $H(f)(V)$ represents the same elements in R as $nH(f)V$, i.e. the set $\{\sum a_i u_i^2 \mid u_i \in R_p\}$ is the same as $\{\sum_{i=1}^m a_i (\sum_{j=1}^n u_{ij}^2) \mid u_{ij} \in R_p\}$. Hence $H(f)(V) = 0$ by (H3) and induction on m . Conversely, let $S = R(\theta)$ where $\theta = -a + (1 + a^2)^{1/2}$ for $a \in R$. θ satisfies the equation $x^2 + 2ax - 1 = 0$. By 4.4, $\text{Ker } H(f_{S/R})$ is torsion. Now $\ker H(f) \subseteq H_t(R)$ follows from induction. $\square$

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