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HYPERBOLIC INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. Hyperbolic integrodifferential equations are defined and conditions sufficient for hyperbolicity are given. The theory includes that of constant coefficient hyperbolic partial differential equations. Other examples are given.

1. **Introduction.** Consider

$$u(x, t) = \sum_{\nu=1}^{p} \int_0^t k_\nu(t-\tau) L_\nu u(x, \tau)\, d\tau + g(x, t) \tag{E}$$

when $x = (x_1, \cdots, x_n)$ and L_ν is a constant coefficient differential operator with respect to these variables.

Definition. (E) is hyperbolic if for each $g \in C(R^n \times [0, \infty)) \cap C^\infty(R^n \times (0, \infty))$ satisfying $g(x, t) = 0$ when $|x_i| \geq b_i$, $i = 1, \cdots, n$, there exists a unique solution $u \in C(R^n \times [0, \infty)) \cap C^\infty(R^n \times (0, \infty))$ having finite signal speed; i.e., there are $c_i \geq 0$ such that u at time t, $t \geq 0$, vanishes outside of $\{-b_i - c_i t < x_i < b_i + c_i t \mid i = 1, \cdots, n\}$.

The above definition parallels the characterization of constant coefficient hyperbolic partial differential equations given in [4]. Moreover, the Cauchy problem for a constant coefficient differential operator with associated polynomial having its degree p equal to its degree in one of its variables can be reduced to (E) by integrating p times with respect to the preferred variable.

Denote the Laplace transform of k_ν by $\tilde{k}_\nu$ and the polynomial associated with L_ν by $Q_\nu(\zeta)$; i.e., $L_\nu = Q_\nu(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. An elementary analysis using both Fourier and Laplace transforms leads to sufficient conditions for hyperbolicity in terms of

$$\sum_{\nu=1}^{p} Q_\nu(i\zeta)\tilde{k}_\nu(w) \Big/ \left(1 - \sum_{\nu=1}^{p} Q_\nu(i\zeta)\tilde{k}_\nu(w)\right). \tag{1.1}$$

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The proof that (E) is hyperbolic if these conditions hold is given in §2. It is an application of the Paley-Wiener theorem. That these conditions are equivalent to the usual hyperbolicity condition when (E) is the integrated form of a partial differential equation is shown in §3. Other examples of hyperbolic equations and an example of what might be called a parabolic equation (the solution has infinite signal speed) are also given in §3.

One possible application of this theory is the classification of equations of fading memory theories in continuum physics. Two of many works that discuss such theories are [3] and [7]. (The techniques of §2 can easily be applied to systems of equations.)

2. **Sufficient conditions.** Denote the Fourier transform of $u(x, t)$ with respect to $x = (x_1, \cdots, x_n)$ by $\hat{u}(\zeta, t)$. (See [5] for basic Fourier transform theory.) If $\hat{u}$ exists,

$$\hat{u}(\zeta, t) - \sum_{\nu=1}^{p} Q_\nu(i\zeta)\int_0^t k_\nu(t-\tau)\hat{u}(\zeta, \tau)\,d\tau = \hat{g}(\zeta, t). \tag{2.1}$$

Solving by successive approximations, we have

$$\hat{u}(\zeta, t) = \hat{g}(\zeta, t) + \int_0^t M(t-\tau, \zeta)\hat{g}(\zeta, \tau)\,d\tau \tag{2.2}$$

where

$$M(t, \zeta) = \sum_{j=1}^{\infty} (-1)^j k^{(j)}(t, \zeta), \tag{2.3}$$

$$k^{(1)}(t, \zeta) = -\sum_{\nu=1}^{p} Q_\nu(i\zeta)k_\nu(t), \tag{2.4}$$

and for $j = 2, \cdots,$

$$k^{(j)}(t, \zeta) = \int_0^t k^{(1)}(t-\tau, \zeta)k^{(j-1)}(\tau, \zeta)\,d\tau. \tag{2.5}$$

Lemma. *$\hat{u}$ is an entire function in ζ.*

Proof. Clearly, $\hat{g}$ is entire and, by (2.3), (2.4) and (2.5), M is entire. Indeed,

$$k^{(j)}(t, \zeta) = \sum_{\nu_1=1}^{p} \cdots \sum_{\nu_j=1}^{p} a_{\nu_1\ldots\nu_j}(t)\, Q_{\nu_1}(i\zeta) \cdots Q_{\nu_j}(i\zeta)$$

where $a_{\nu_1\ldots\nu_j}(t) = O(1/(j-1)!)$. Consequently, (2.3) converges absolutely for each ζ and the series can be rearranged as a power series. Since $\hat{g}$ and M are entire, u is entire.

Assume that $k_\nu(t)$ is piecewise continuous and that for each ν,

(a) $|k_\nu(t)| \leq Me^{at}$.

The existence of $\tilde{M}(w, \zeta)$ for re w greater than some w_1 then follows from (2.3). Moreover, the kernels $k^{(1)}$ and M satisfy the reciprocal relation

$$-\sum_{\nu=1}^{p} Q_\nu(i\zeta)k_\nu(t) + M(t, \zeta) = \sum_{\nu=1}^{p} Q_\nu(i\zeta)\int_0^t k_\nu(t-\tau)M(\tau, \zeta)\,d\tau$$

and, consequently, $\tilde{M}(w, \zeta)$ is equal to (1.1). M can be recovered from the inversion formula

$$M(t, \zeta) = \frac{1}{2\pi i}\int_{w_2 - i\infty}^{w_2 + i\infty} e^{tw}\tilde{M}(w, \zeta)\,dw$$

if $w_2 > w_1$. (The Laplace transform theory referred to can be found in [1] or [6].)

Assume that for each ζ, $\tilde{M}(w, \zeta)$ is an analytic function with at most a finite number $s(\zeta)$ of singularities $w_j(\zeta)$. Assume that these singularities are either poles or essential singularities of the first kind and that

(b) $\sup s(\zeta) = s < \infty$,

(c) re $w_j(\zeta) = O(|\zeta|)$ as $|\zeta| \to \infty$, and

(d) re $w_j(\zeta)$ is bounded for $\zeta \in R^n$.

Let $C(w, \delta)$ be a circle of radius δ about w. The residue of $e^{tw}\tilde{M}(w, \zeta)$ at $w_j(\zeta)$ is

$$\frac{1}{2\pi i}\exp(w_j(\zeta)t)\int_{C(0,\,\delta(\zeta, j))} e^{wt}\tilde{M}(w + w_j(\zeta), \zeta)\,dw$$

for $\delta(\zeta, j)$ sufficiently small. Since $\tilde{M}(w, \zeta) \to 0$ as $|w| \to \infty$,

$$(2.6)\quad M(t, \zeta) = \frac{1}{2\pi i}\sum_{j=1}^{s(\zeta)} \exp(w_j(\zeta)t)\int_{C(0,\delta(\zeta,j))} e^{tw}\tilde{M}(w + w_j(\zeta), \zeta)\,dw.$$

Finally, assume that there exists $r \geq 0$ such that

$$\text{(e)}\quad \sup_{\substack{\zeta=(\zeta_1,\cdots,\zeta_n)\\ j=1,\cdots,s(\zeta)}} \frac{1}{(1+\Sigma|\zeta_i|^2)^r}\left|\int_{C(0,\delta(\zeta,j))} e^{tw}\tilde{M}(w + w_j(\zeta), \zeta)\,dw\right| \leq C.$$

We now apply the

Paley-Wiener theorem. *If an entire analytic function $f(\zeta)$ satisfies*

$$|f(\zeta)| \leq C_\epsilon \exp\sum_{i=1}^{n}(b_i + \epsilon)|\zeta_i|$$

for any $\epsilon > 0$ *and if* $f(\eta)$, $\eta \in R^n$, *is square integrable, then* f *is the Fourier transform of a function* $h(x) \in L^2(R^n)$ *which vanishes a.e. outside of* $G_b = \{-b_j \leq x_j \leq b_j,\ j = 1, \cdots, n\}$.

(A proof of this theorem in an equivalent form can be found in [2].) Since g is infinitely differentiable and vanishes outside of G_b, for all $t \geq 0$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, α_i nonnegative integers,

$$|\zeta^\alpha \hat{g}(\zeta, t)| \leq C \exp \sum_{i=1}^{n} b_i |\zeta_i| \tag{2.7}$$

and $\eta^\alpha \hat{g}(\eta, t)$, $\eta \in R^n$, is square integrable. ($\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$.) Clearly, (2.2), (2.6), (2.7), (b), (c) and (e) imply that

$$|\hat{u}(\zeta, t)| \leq C \exp \sum_{i=1}^{n} (b_i + ct)|\zeta_i|,$$

c independent of t. Moreover, (2.2), (2.6), the square integrability of $\eta^\alpha \hat{g}(\eta, t)$ for any α, (b), (d) and (e) imply that for each $t > 0$, $\hat{u}(t, \eta) \in L^2(R^n)$. By the Paley-Wiener theorem $\hat{u}(\cdot, t)$ is the Fourier transform of a function in $L^2(R^n)$ vanishing a.e. outside of $\{-b_j - ct \leq x_j \leq b_j + ct\}$. To show this function is a solution of (E), we apply the same analysis to

$$v(x, t) = \sum_{\nu=1}^{p} \int_0^t k_\nu(t - \tau) L_\nu v(x, \tau)\, d\tau + D^\alpha g(x, t)$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

This leads to $\hat{v} = (i\zeta)^\alpha \hat{u}(\zeta, t)$ being the Fourier transform of a function in $L^2(R^n)$ and, hence, in $L^2(R^n)$ itself. This being true for all α implies, by Sobolev's lemma, $u(x, t) \in C^\infty(R^n)$ for each t. Existence now follows in a straightforward manner. Uniqueness is immediate. (That u is C^∞ in t follows from (2.2) and the inverse transform formula.)

3. **Examples.** We first show that conditions (a)–(e) hold if (E) arises from the Cauchy problem for the hyperbolic partial differential equation $P(\partial/\partial t, \partial/\partial x_1, \cdots, \partial/\partial x_n)u = f$ when

$$P(t, \zeta) = t^p + \sum_{\nu=1}^{p} Q_\nu(\zeta) t^{p-\nu},$$

$|Q_\nu(\zeta)| \le C(|\zeta|^\nu + 1)$. P is hyperbolic if and only if the roots $w(\zeta)$ of $P(w, i\zeta) = 0$ have bounded real part for $\zeta \in R^n$. Integrating p times with respect to t, we have (E) when $k_\nu(t) = -t^{\nu-1}/(\nu-1)!$. Moreover,

$$\tilde{M}(w, \zeta) = -\sum_{\nu=1}^{p} w^{p-\nu} Q_\nu(i\zeta)/P(w, i\zeta).$$

Clearly, (a)–(e) hold. ((c) follows from the estimate given in [5, p. 148].)

Consider (E) when $k_\nu(t) = a_\nu t^{\alpha_\nu - 1} e^{\beta_\nu t}$, α_ν a positive integer. It follows that $\tilde{M}(w, \zeta) = P_1(w, i\zeta)/P_2(w, i\zeta)$ where

$$P_1(w, i\zeta) = \sum_{\nu=1}^{p}\left(\prod_{\mu=1;\, \mu\neq\nu}^{p} (w - \beta_\mu)^{\alpha_\mu}\right) a_\nu(\alpha_\nu - 1)! Q_\nu(i\zeta)$$

and

$$P_2(w, \zeta) = \prod_{\mu=1}^{p} (w - \beta_\mu)^{\alpha_\mu} - P_1(w, \zeta).$$

If $P_2(w, \zeta)$ is a hyperbolic polynomial, (E) is hyperbolic. In particular, if $p = 1$, L_1 is a homogeneous second order elliptic operator and

$$k_1(t) = \alpha_1 - \alpha_1 \exp(-\alpha_2 t) + \alpha_3 t, \qquad \alpha_i > 0,$$

then (E) is hyperbolic. This example has the additional property that re $w_j(\zeta) < 0$ for $\zeta \in R^n - \{0\}$. Equations with this property, the property that re $w_j(\zeta) = 0$ for all j and ζ or some intermediate property, probably should be singled out in a physical theory. The equation

$$u(x, t) = \int_0^t e^{\alpha(t-\tau)} \sin(t-\tau)\Delta u(x, \tau)\, d\tau + g(x, t)$$

is hyperbolic but

$$u(x, t) = \int_0^t e^{\alpha(t-\tau)} \cos(t-\tau)\Delta u(x, \tau)\, d\tau + g(x, t)$$

is not. $\tilde{M}(w, \zeta)$ is

$$\frac{-|\zeta|^2}{|\zeta|^2 + 1 + (w-\alpha)^2} \quad \text{and} \quad \frac{-|\zeta|^2(w-\alpha)}{(w-\alpha)^2 + (w-\alpha)|\zeta|^2 + 1},$$

respectively. The denominator of $\tilde{M}(w, \zeta)$ is a hyperbolic polynomial in the first case and a parabolic polynomial in the second case. This means that the real part of the roots approach $-\infty$ as $|\zeta| \to \infty$ through real values. Consequently, the solution u will exist as a C^∞ function having a Fourier transform which is $O(\exp t|\zeta|^2)$, implying that u has infinite signal speed.

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