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THE USE OF ATTRACTIVE FIXED POINTS IN ACCELERATING THE CONVERGENCE OF LIMIT-PERIODIC CONTINUED FRACTIONS[1]

JOHN GILL

ABSTRACT. A continued fraction can be interpreted as a composition of Möbius transformations. Frequently these transformations have powerful attractive fixed points which, under certain circumstances, can be used as converging factors for the continued fraction. The limit of a sequence of such fixed points can be employed as a constant converging factor.

The continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots \tag{1}$$

is said to be *periodic in the limit* provided $\lim a_n = a$ and $\lim b_n = b \neq 0$.

Set

$$t_n(z) = a_n/(b_n + z), \qquad n \geq 1,$$

$$T_1(z) = t_1(z), \qquad T_n(z) = T_{n-1}(t_n(z)), \qquad n \geq 2,$$

and

$$\lim t_n(z) = t(z) = a/(b + z).$$

The nth approximant of (1) is obtained by setting $z = 0$ in

$$T_n(z) = \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n + z} . \tag{2}$$

(1) is called *periodic* if $t_n(z) \equiv t(z)$, $n \geq 1$; and if t has two distinct fixed points, u and v, where $|u|/|v| < 1$, then one can write [1]

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(3) $$(T_n(z) - u)/(T_n(z) - v) = (u/v)^n (z - u)/(z - v), \qquad n \geq 1.$$

Clearly, $\lim T_n(z) = u$ for $z \neq v$.

An exact truncation error for a fixed z follows easily from (3). For $z = 0$ and $z = u$ this takes the forms

(4) $$T_n(0) - u = -uK^n(1 - K)/(1 - K^{n+1}), \qquad n \geq 1,$$

and

(5) $$T_n(u) - 0 \equiv 0, \qquad n \geq 1,$$

where $K = u/v$.

Instant maximum acceleration of the periodic fraction occurs, therefore, upon replacing $z = 0$ by $z = u$ in (2). Let $\{u_n\}$ and $\{v_n\}$ be the fixed points of $\{t_n\}$, chosen so that $|u_n|/|v_n| < 1$. This paper is devoted to describing certain continued fractions, periodic in the limit, whose convergence may be speeded by setting $z = u$ or $z = u_{n+1}$ in $T_n(z)$. A geometrical approach leads to a priori truncation error estimates of $T_n(u)$ and $T_n(u_{n+1})$. The technique is similar to that used in [2].

Previous articles associating the convergence behavior of continued fractions with the behavior of the sequences $\{u_n\}$ and $\{v_n\}$ include [2], [3] and [6]. Papers concerned with converging factors and/or contraction maps include [4] and [7].

Computations involving u (or u_{n+1}) are accomplished as follows: Let P_n and Q_n be the nth partial numerator and nth partial denominator of (1) [5] so that $T_n(0) = P_n/Q_n$, $n \geq 1$. Let $P_n^* = P_n + uP_{n-1}$, $Q_n^* = Q_n + uQ_{n-1}$, $n \geq 2$. Then $T_n(u) = P_n^*/Q_n^*$, $n \geq 2$.

The phenomena of instantaneous convergence is not restricted to periodic fractions. Write t_n in terms of its fixed points.

(6) $$t_n(z) = -u_n v_n/[-(u_n + v_n) + z], \qquad n \geq 1.$$

Let $u_n \equiv u$, $\lim v_n = v$. Then (1) becomes

(7) $$\frac{uv_1}{u + v_1} - \frac{uv_2}{u + v_2} - \cdots - \frac{uv_n}{u + v_n} - \cdots$$

Theorem 1. *Let T_n be defined in accordance with* (6). *If* $0 < |u| < |v_n|$, $n \geq 1$ *and* $|u| < |v|$, *then* $\lim T_n(0) = \lim T_n(u) = T_n(u) \equiv u$.

Proof. Theorem 1 [6] implies $\{T_n(z)\}$ converges to a common limit for all $z \neq v$. $T_n(u) \equiv u$, since $t_n(u) = u$, $n \geq 1$.

Theorem 2. *Let* T_n *be defined as before. If* $u = v$ *and* $\Sigma|v_n - v_{n+1}| < \infty$, *then* $\lim_n T_n(0) = \lim_n T_n(u) = T_n(u) \equiv u$.

Proof. Theorem 1 [3] guarantees the convergence of $\{T_n(z)\}$ to a common limit for every z.

It is well known that (1) converges provided $\lim(|u_n|/|v_n|) = |u|/|v| < 1$. $\lim T_n(0)$ is near u if $u_n \approx u$, $v_n \approx v$. The pattern of convergence is more complicated when $u = v$ or when $u \neq v$, but $|u| = |v|$. The first of these two special cases occurs in Theorem 2.

In the three theorems that follow it is assumed that $u_n \to u \neq 0$, $v_n \to v \neq 0$, $|u_n| < |v_n|$ and $|u| < |v|$, even though the last two restrictions may not always be necessary. Although having a formidable appearance, the hypotheses are not too difficult to satisfy.

Set

$$\epsilon_n = |u_n - u| \cdot |u_n|/[|v_n| - |u_n - u|], \qquad n \geq 1, \tag{8}$$

and $H_n = |u_{n+1} - u_n| + \epsilon_{n+1}$.

Theorem 3. *If*

(i) $|u_n - u| > |u_n - u_{n+1}|$,

(ii) $|v_n| > 2|u_n - u|$,

(iii) $|v_n| \geq |u_n - u| + |u_n - u| \cdot |u_n|/[|u_{n-1} - u| - |u_n - u_{n-1}|]$,

(iv) $|v_n| > |u_{n+1} - u_n| + \epsilon_{n+1}$

are all satisfied, then $\lim T_n(u) = \lim T_n(u_{n+1}) = \lim T_n(0) = T$, *where* $|T - u_1| \leq \epsilon_1$. *Furthermore,*

$$2\epsilon_n \prod_{1}^{n-1} \{|u_j v_j|[|u_{j+1} - u_j - v_j| - \epsilon_{j+1}]^{-2}\} \tag{9}$$

is an upper bound on both the errors $|T_n(u) - T|$ *and* $|T_{n-1}(u_n) - T|$. *The monotonic divergence to* 0 *of the product is guaranteed by*

(v) $|v_n|^2 - |v_n|(|u_n| + 2H_n) + H_n^2 > 0, \qquad n \geq 1.$

Proof of Theorem 3. $|K_n| = |u_n/v_n| < 1$, $|K| < 1$ imply $\lim T_n(z) = \lim T_n(0) = T$ for $z \neq v$ [6, Theorem 1]. The boundary of the disk

$$C(\epsilon_n, u_n) = \{z : |z - u_n| \leq c_n |z - u_n - v_n|\} = \{z : |t_n(z) - u_n| \leq \epsilon_n\},$$

with ϵ_n defined by (8), and

$$0 < c_n = \epsilon_n/|u_n| < 1 \tag{10}$$

is a circle of Appollonius with respect to the points u_n and $u_n + v_n$ (see the figure). Its center, g_n, and radius, R_n, are given by

$$g_n = u_n - c_n^2 v_n (1 - c_n^2)^{-1}, \tag{11}$$

and

$$R_n = c_n |v_n| (1 - c_n^2)^{-1}. \tag{12}$$

The following conditions hold.

(1) $t_n(C(\epsilon_n, u_n)) = N(\epsilon_n, u_n) = \{z: |z - u_n| \leq \epsilon_n\}, \; n \geq 1.$

(2) $u \in C(\epsilon_n, u_n), \; n \geq 1.$

(3) $N(\epsilon_n, u_n) \subset C(\epsilon_{n-1}, u_{n-1}), \; n \geq 2.$

(1) follows from (8), (ii) and (10).

(2) is equivalent to $|g_n - u| \leq R_n, \; n \geq 1$, which, by (11) and (12) can be written

$$|u_n - u - c_n^2 v_n (1 - c_n^2)^{-1}| \leq c_n |v_n| (1 - c_n^2)^{-1}.$$

This inequality follows from

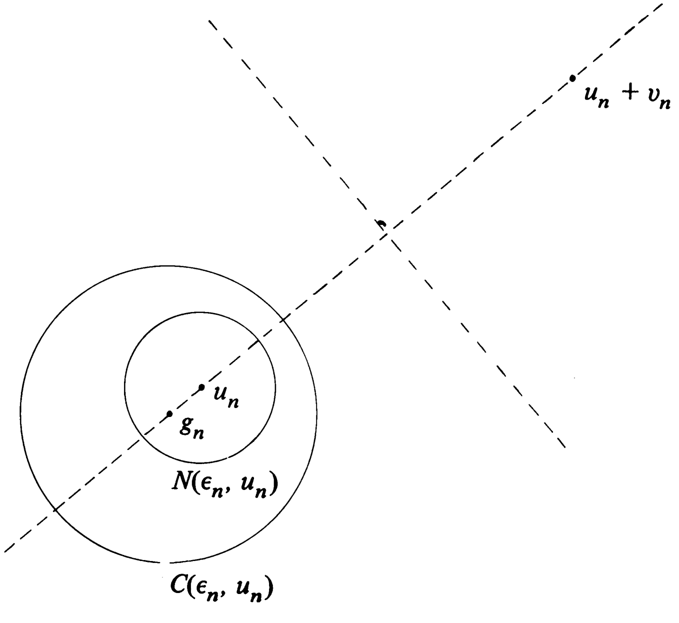
$$|u_n - u| + c_n^2 |v_n| (1 - c_n^2)^{-1} = c_n |v_n| (1 - c_n^2)^{-1},$$

which is equivalent to (8), if we assume (ii).

(3) can be written $|g_{n-1} - u_n| + \epsilon_n \leq R_{n-1}, \; n \geq 2$, which will follow if

$$|u_{n-1} - u_n| + c_{n-1}^2 |v_{n-1}| (1 - c_{n-1}^2)^{-1} + \epsilon_n \leq c_{n-1} |v_{n-1}| (1 - c_{n-1}^2)^{-1}.$$

(i) and (ii) suffice to imply the equivalence of this last inequality to (iii).



The contractive property of the t_n's now come into play. From (6) and the figure

$$|t_n(z) - t_n(z')| = \frac{|u_n v_n| \cdot |z - z'|}{|z - (u_n + v_n)| \cdot |z' - (u_n + v_n)|}$$

$$\leq \frac{|u_n v_n|}{(|u_{n+1} - u_n - v_n| - \epsilon_{n+1})^2} \cdot |z - z'|,$$

where z and $z' \in N(\epsilon_{n+1}, u_{n+1})$. (iv) insures $|u_{n+1} - u_n - v_n| > \epsilon_{n+1}$. Thus the bound (9) is obtained. (v) makes the factors of the product less than one. The limit of these factors as $j \to \infty$ is $|u|/|v| < 1$.

Let $\{\zeta_n\}$ be any sequence of points such that $\zeta_n \in N(\epsilon_n, u_n)$. Then $\lim T_{n-1}(\zeta_n) = \lim T_n(0) = T$, and $|T_{n-1}(\zeta_n) - T|$ is bounded by (9). An obvious choice of $\{\zeta_n\}$ is $\{u_n\}$.

This completes the proof of Theorem 3.

The hypotheses of Theorem 3 can be altered slightly to produce an alternate and possibly weaker form of (9).

Set

$$\epsilon_n = |u_n - u|, \qquad n \geq 1, \tag{13}$$

and

$$l_n = |u_{n+1} - u_n| + \epsilon_{n+1}, \qquad n \geq 1.$$

Theorem 4. *If*

(i) $0 < \epsilon_n < |u_n|$, $n \geq 1$,

(ii) $|v_n| > |u_n| + \epsilon_n$, $n \geq 1$,

(iii) $|v_n| \geq (1 + |u_n|\epsilon_n^{-1}) l_n$, $n \geq 1$,

are all satisfied, then $\lim T_n(u) = \lim T_n(u_{n+1}) = \lim T_n(0) = T$, *where* $|T - u_1| \leq \epsilon_1$. *Furthermore,*

$$2\epsilon_n \prod_{1}^{n-1} \{|u_j v_j|[|u_{j+1} - u_j - v_j| - \epsilon_{j+1}]^{-2}\} \tag{14}$$

is an upper bound on both the errors $|T_n(u) - T|$ *and* $|T_{n-1}(u_n) - T|$. *The monotonic divergence to* 0 *of the product is guaranteed by*

(iv) $|v_n|^2 - |v_n|(|u_n| + 2l_n) + l_n^2 > 0$, $n \geq 1$.

Remark. If $\epsilon_n(\downarrow)$ and $|u_{n+1} - u_n| < \epsilon_n$, then $|v_n| > 2(\epsilon_n + |u_n|)$ implies (iii).

Proof of Theorem 4. Conditions (1), (2), and (3) in the proof of Theorem 3 are satisfied assuming (i), (ii) and (iii) of Theorem 4. The details of proof are much the same.

Example 1. $[300(10^{-4n}-1)/(20+10^{1-4n})i]_{n=1}^{\infty}$, where $u_n = 10(1-10^{-4n})i \longrightarrow u = 10i$ and $v_n \equiv v = -30i$. The hypotheses of Theorem 3 are satisfied.

actual error:	$\lvert T_n(0) - T\rvert$	$\lvert T_n(u) - T\rvert$	$\lvert T_n(u_{n+1}) - T\rvert$	*error bound*
$N = 2$	1.4	1.4×10^{-12}	3.7×10^{-15} ↘	2.2×10^{-9}
$N = 3$	5×10^{-1}	2.1×10^{-15}		7.3×10^{-14}

Note that the error bound for $\lvert T_2(u_3) - T\rvert$ is given in the second row of figures.

The hypotheses of Theorem 3 can be changed, in the following special case without affecting the error bound (9). Replace b_n by 1 in (1).

Set

(15) $$\epsilon_n = \lvert u_n\rvert(u_n - u)/(u+1), \qquad n \geq 1.$$

Theorem 5. *If*

(i) u_n *and* $u \neq -1$, -1 *lie on a ray, and* u *is between* u_n *and* -1,

(ii) $\lvert u_n - u\rvert\ (\downarrow)$,

(iii) $\lvert u_n - u\rvert < \lvert u-1\rvert$,

(iv) $\lvert u_n\rvert < \lvert u+1\rvert$

are all satisfied, then $\lim T_n(u) = \lim T_n(u_{n+1}) = \lim T_n(0) = T$, *where*

$$\lvert T - u_1\rvert \leq \lvert u_1\rvert \cdot \lvert u_1 - u\rvert / \lvert 2u_1 - u + 1\rvert.$$

Furthermore,

(16) $$2\epsilon_n \prod_1^{n-1} \{\lvert u_j(u_j+1)\rvert[\lvert u_{j+1}+1\rvert - \epsilon_{j+1}]^{-2}\}$$

is an upper bound on both the errors $\lvert T_n(u) - T\rvert$ *and* $\lvert T_{n-1}(u_n) - T\rvert$.

Proof. Observe that $v_n = -u_n - 1$ and $v = -u - 1$.

(1) follows from (i) and (iii).

(2) is satisfied if $\lvert g_n - u\rvert = R_n$. A brief computation shows that this is equivalent to (15). (ii) implies $\lvert g_{n-1} - u\rvert \geq \lvert g_n - u\rvert$, which guarantees $C(\epsilon_n, u_n) \subset C(\epsilon_{n-1}, u_{n-1})$. (iv) is equivalent to $N(\epsilon_n, u_n) \subset C(\epsilon_n, u_n)$. Hence (3) is satisfied.

Example 2. Set $F(x) = (x/\text{Arctan}\ x) - 1$, where [5, p. 116]

$$\text{Arctan}\, x = \frac{x}{1} + \frac{x^2}{3} + \cdots + \frac{n^2x^2}{2n+1} + \cdots .$$

An equivalence transformation on F gives

$$F(x) = \left[\frac{n^2x^2/(4n^2-1)}{1}\right]_{n=1}^{\infty} .$$

The hypotheses of Theorem 5 are satisfied for $x = \sqrt{3}$.

actual error:	$\lvert T_n(0) - T\rvert$	$\lvert T_n(u) - T\rvert$	$\lvert T_n(u_{n+1}) - T\rvert$	*error bound* (16)
$n = 2$	9.8×10^{-2}	1.8×10^{-3}	2.1×10^{-4}	7.5×10^{-3}
$n = 3$	3.5×10^{-2}	3.3×10^{-4}	3.1×10^{-5}	1.1×10^{-3}
$n = 4$	1.1×10^{-2}	7.0×10^{-5}		2.1×10^{-4}

Although the theorems in this paper can be applied to any type continued fraction, frequently the fixed points have complicated structures. The following formal procedure can be used to convert a power series into a "fixed point type" fraction.

Let $P(z) = 1 + a_1^{(1)}z + a_2^{(1)}z^2 + \cdots$. Write

$$P(z) = 1 + c_1 z^{k_1}(1 + b_{n_1}^{(1)} z^{n_1} + \cdots), \qquad k_1 \geq 1,\ n_1 \geq 1,$$

$$c_1 = a_{k_1}^{(1)}, \qquad k_1 = \min\{n: a_n^{(1)} \neq 0\}.$$

Then

$$P(z) = 1 + \frac{c_1 z^{k_1}}{1 + a_1^{(2)}z + a_2^{(2)}z^2 + \cdots}$$

$$= 1 + \frac{c_1 z^{k_1}}{1 + c_1 z^{k_1} - c_2 z^{k_2}(1 + b_{n_2}^{(2)} z^{n_2} + \cdots)}, \qquad \text{etc.}$$

This gives rise to the fraction

$$1 - \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{-c_n z^{k_n}}{1 + c_n z^{k_n}}\right), \tag{17}$$

with fixed points $u_n = c_n z^{k_n}$, $v_n \equiv 1$. Observe that (17) appears in the equivalent continued fraction expansion of

$$Q(z) = 1 + (2 - P(z))^{-1} = 1 + 1 + c_1 z^{k_1} + \cdots .$$

A purely formal application to e^x, $x = 1/10$, gives

	actual error: $\lvert T_n(u_{n+1}) - e^{1/10}\rvert$
$n = 2$	5×10^{-7}
$n = 3$	1×10^{-8}

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DEPARTMENT OF MATHEMATICS, SOUTHERN COLORADO STATE COLLEGE, PUEBLO, COLORADO 81005