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A GENERALIZED SHILOV BOUNDARY AND ANALYTIC STRUCTURE[1]

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ABSTRACT. A generalization of the concept of the Shilov boundary of a uniform algebra is introduced. This makes it possible to formulate and prove several-dimensional analogues of certain well-known results which guarantee the existence of one-dimensional analytic structure when a function in the algebra is finite-to-one over a suitable part of its spectrum.

A major problem in the study of uniform algebras is to find interesting sufficient conditions for the existence of analytic structure in their maximal ideal spaces; see, e.g., [6, Chapter 3 and §30]. With one notable exception (a result due to Gleason [3], or see [6, Theorem 15.2]), most of these results are one dimensional. In this paper we indicate how one kind of condition ([7, Theorems 10.7 and 11.2], which were derived from Bishop's paper [2]) can be generalized to yield several-dimensional analytic structure. First we must define a generalization of the Shilov boundary of a uniform algebra.

Notation. A will always denote a uniform algebra defined on the compact Hausdorff space X with maximal ideal space M. $\partial_0 A$ is the usual Shilov boundary of A. Let

$$A^n = \{(f_1, \cdots, f_n) \mid f_1, \cdots, f_n \in A\},$$

so that each $F = (f_1, \cdots, f_n) \in A^n$ maps M to $\mathbf{C}^n$, and $F(M)$ is the joint spectrum of $f_1, \cdots, f_n$. If K is a compact subset of M, let

$$A_K = \{f \in C(K) \mid f \text{ is a uniform limit on } K \text{ of functions from } A\}.$$

If $F \in A^n$, let

$$V(F) = \{x \in M \mid F(x) = 0\}$$

be the A-variety corresponding to F and note that $V(F)$ is A-convex, i.e., the maximal ideal space of $A_{V(F)}$ is $V(F)$.

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If Y is a topological space, and if $Z \subseteq Y$, then $\partial_Y Z$, or simply ∂Z, will denote the topological boundary of Z relative to Y. For $z \in \mathbf{C}^n$ we let $|z| = (\sum_{j=1}^n |z_j|^2)^{1/2}$, the usual Euclidean norm of z, and then we define

$$B^n = \{z \in \mathbf{C}^n \,|\, |z| < 1\}, \qquad S^n = \{z \in \mathbf{C}^n \,|\, |z| = 1\}.$$

Finally, m_k will denote k-dimensional Lebesgue measure; its domain will always be clear from the context.

Definition. $\partial_n A = \text{Closure}[\bigcup\{\partial_0(A_{V(F)}) \,|\, F \in A^n\}]$.

(At this point the reader may wish to look ahead to the statements of Theorems 1 and 2 to see the direction in which our development will proceed.)

Lemma 1. *Let $x \in K \subseteq M$, K closed. Let $F \in A^n$ and suppose that $x \in V(F)$. Then $\forall g \in A_{V(F)}$,*

$$|g(x)| \leq \max\{|g(y)| \,|\, y \in [\partial K \cap V(F)] \cup [\partial_n A \cap V(F) \cap K]\}.$$

Proof. This follows at once from the preceding definition and the usual local maximum modulus principle [7, Theorem 9.3] applied to $A_{V(F)}$.

Corollary 1. *Let K be a compact A-convex subset of $M \backslash \partial_n A$. Then $\partial_n(A_K) \subseteq \partial K$.*

Proof. Suppose that the Corollary is false, so that $\partial_n(A_K) \not\subseteq \partial K$. Then there exists $G = (g_1, \cdots, g_n) \in (A_K)^n$ such that the interior of K meets the Shilov boundary of $(A_K)_{V(G)}$. Consequently there is an $h \in A$ and a $p \in \text{int}(K) \cap V(G)$ such that $|h(p)| > 1$, but $|h| < 1$ on $\partial K \cap V(G)$. Now choose $\epsilon > 0$ such that $|h| < 1$ on $\{x \in \partial K \,|\, |g_j(x)| < \epsilon,\ 1 \leq j \leq n\}$. Let $f_1, \cdots, f_n \in A$ be chosen so that $f_j(p) = 0$, while $|f_j - g_j| < \epsilon$ on K, $1 \leq j \leq n$. Then $|h| < 1$ on $\partial K \cap V(f_1, \cdots, f_n)$, $p \in V(f_1, \cdots, f_n)$, and $|h(p)| > 1$. This contradicts the local maximum modulus principle of Lemma 1, so the Corollary is established.

Lemma 2. (Cf. [7, Lemma 11.1].) *Let $n > 0$ and suppose that $\partial_{n-1} A \subseteq X$. Let $F \in A^n$ and let W be a component of $\mathbf{C}^n \backslash F(X)$. Then either $F(M) \cap W = \emptyset$ or $F(M) \cap W = W$.*

Proof. Suppose $\emptyset \neq F(M) \cap W \neq W$. Then $\exists x \in M$ such that $z^1 = F(x) \in \partial[F(M)] \cap W$. Choose $z^0 \in W \backslash F(M)$ such that $|z^0 - z^1| < \alpha = \min\{|z - z^0| \,|\, z \in F(X)\}$. Wlog, $z^0 = 0$, $z^1 = (1, 0, \cdots, 0)$; so $\alpha > 1$.

Let $F = (f_1, \cdots, f_n)$. Since $0 \notin F(M)$, $\exists h_1, \cdots, h_n \in A$ such that $\sum_{j=1}^n h_j f_j = 1$ (see [7, Lemma 8.1]). Let $G = (f_2, \cdots, f_n) \in A^{n-1}$. Since $x \in V(G)$, we have

$$|h_1(x)| \le \max_{V(G)\cap\partial_{n-1}A} |h_1| = \max_{V(G)\cap X} |h_1|$$

$$= \max_{V(G)\cap X} \frac{1}{|f_1|} = \max_{V(G)\cap X} \frac{1}{|F|} \le 1/\min_X |F| = \frac{1}{\alpha} < 1.$$

But also

$$h_1(x) = \frac{1 - \sum_{j=2}^n h_j(x) f_j(x)}{f_1(x)} = 1.$$

This contradiction shows that we must originally have had $F(M) \cap W = \emptyset$ or $F(M) \cap W = W$.

Theorem 1. *Let $n > 0$ and let $\partial_{n-1}A \subseteq X$. Let $F \in A^n$ and suppose that $|F| = 1$ on X, that $0 \in F(M)$, and that $\exists S' \subseteq S^n$ such that $m_{2n-1}(S') > 0$ and $\forall \lambda \in S'$ $\exists$ a unique $q \in X$ with $F(q) = \lambda$. Then*

$\forall \zeta \in B^n$ $\exists$ a unique $x \in M$ with $F(x) = \zeta$;

$\forall f \in A$, $f \circ F^{-1}$ is holomorphic on B^n.

Proof. The case $n = 1$ is Theorem 10.7 in [7], so we assume that $n > 1$. Let $z \in B^n$. Since $m_{2n-1}(S') > 0$, it follows that for some complex line $L = \{z + \zeta\lambda | \zeta \in \mathbf{C}\}$ (where $\lambda \in \mathbf{C}^n \setminus \{0\}$) through z we have $m_1(S' \cap L) > 0$. For simplicity assume that $\lambda = (1, 0, \cdots, 0)$ and that $z = 0$, and let $G = (f_2, \cdots, f_n)$. One readily verifies that $f_1|_{V(G)}$ is a function in the uniform algebra $A_{V(G)}$ to which the $n = 1$ result can be applied. (Note that since $0 \in F(M)$, Lemma 2 implies that $F(M) \supseteq B^n$, so that we are justified in assuming that $z = 0$.) We can thus conclude that $\forall z' \in B^n \cap L$, $\exists$ precisely one $x' \in V(G)$ with $F(x') = z'$. Thus $\forall z \in B^n$, $\exists$ a unique $x \in M$ with $F(x) = z$.

Fix ρ, $0 < \rho < 1$. Let

$$K = \{x \in M \,|\, |F(x)| \le \rho\}.$$

By Corollary 1, $\partial_{n-1}(A_K) \subseteq \partial K = \{x \in M | \, |F(x)| = \rho\}$. Since F is one-to-one on K, the result for $n = 1$ implies that $\forall f \in A$, $f \circ F^{-1}$ is holomorphic on the intersection of any complex line with $\{z \in \mathbf{C}^n | \, |z| < \rho\}$. Since ρ was arbitrary, it follows that $\forall f \in A$, $f \circ F^{-1}$ is holomorphic on B^n.

Lemma 3. (Cf. [1, Lemma 3].) *Let $n > 0$ and let $\partial_{n-1}A \subseteq X$. Let $F \in A^n$, let W be a component of $\mathbf{C}^n \setminus F(X)$, let $z \in W$, and suppose that J is a component of $F^{-1}(z)$. Then for each neighborhood $\mathcal{O}$ of J, $F(\mathcal{O})$ is a neighborhood of z; given such an $\mathcal{O}$, $\exists$ a compact A-convex neighborhood N of J such that $N \subseteq \mathcal{O}$ and $z \notin F(\partial N)$.*

Proof. Choose compact sets J', J'' such that $F^{-1}(z) = J' \cup J''$, $J' \cap J'' = \emptyset$, and $J \subseteq J' \subseteq \mathcal{O}$ (cf. [6, Lemma 8.13]). $K = F^{-1}(z)$ is A-convex, so $\exists\, g \in A_K$ such that $g = 0$ on J', $g = 1$ on J'' (by the Shilov idempotent theorem [7, Theorem 8.6]). Choose $h \in A$ with $\max_K |h - g| < \frac{1}{4}$ and define

$$U = \{x \in \mathcal{O} \mid |h(x)| < \tfrac{1}{4}\} \cup \{x \in M \mid |h(x)| > \tfrac{3}{4}\},$$

so that U is a neighborhood of $F^{-1}(z)$. Choose $\epsilon > 0$ such that $\{x \in M \mid |F(x) - z| \leq \epsilon\} \subseteq U$. Let

$$N = \{x \in M \mid |F(x) - z| \leq \epsilon,\ |h(x)| \leq \tfrac{1}{4}\}.$$

Then N is a compact A-convex neighborhood of J and $N \subseteq \mathcal{O}$. If $x \in \partial N$, either $|F(x) - z| = \epsilon$ or $|h(x)| = \frac{1}{4}$, so that $z \notin F(\partial N)$. Finally, $N \subseteq M \backslash \partial_{n-1} A$, so Corollary 1 implies that $\partial_{n-1}(A_N) \subseteq \partial N$. Since $z \in F(N) \backslash F(\partial N)$, Lemma 2 implies that $F(N)$ is a neighborhood of z.

Theorem 2. *Let $n > 0$ and let $\partial_{n-1} A \subseteq X$. Let $F \in A^n$ and let W be a component of $\mathbf{C}^n \backslash F(X)$. Assume that $F(M) \cap W \neq \emptyset$. Suppose $\exists\, W' \subseteq W$ such that $m_{2n}(W') > 0$ and $\forall z \in W'$,*

$$\# F^{-1}(z) = (\text{number of } x \in M \text{ with } F(x) = z)$$

is finite. For $l = 1, 2, \cdots$, let

$$W_l = \{z \in W \mid \# F^{-1}(z) = l\}.$$

Then $\exists$ a positive integer k such that

(i) $W = \bigcup_{j=1}^{k} W_j$;

(ii) $\bigcup_{j=1}^{k-1} W_j$ *is a proper analytic subvariety of* W;

(iii) $F : F^{-1}(W) \to W$ *is a (branched) analytic cover (in the sense of* [5, *p.* 101]).

Consequently, $\exists$ a sheaf $\mathcal{O}$ of germs of functions on $F^{-1}(W)$ such that $(F^{-1}(W), \mathcal{O})$ is an analytic space of pure dimension n and $\forall f \in A$, f is holomorphic on $F^{-1}(W)$ (for definitions, see [5, *pp.* 147–155]).

Proof. We shall assume that $n > 1$, as the $n = 1$ case is essentially Theorem 11.2 in [7]. (If X is not metric, any difficulty with the measurability of sets needed in the proof of Theorem 11.2 can be dealt with by the kind of idea developed in Assertion 1 below.) For $l = 1, 2, \cdots$, let $V_l = \bigcup_{j=1}^{l} W_j$.

Assertion 1. *If T is a relatively closed subset of W and if $\forall z \in T$, $\# F^{-1}(z)$ is finite, then $\forall l$, $V_l \cap T$ is relatively closed and hence $W_l \cap T$ is measurable.*

Proof of Assertion 1. Suppose $z \in T$, $z_n \in V_l \cap T$, $z_n \to z$. Let $m = \#F^{-1}(z) < \infty$. By Lemma 3 $\exists$ disjoint compact sets $N_1, \cdots, N_m$ such that $F(N_j)$ is a neighborhood of z for $j = 1, \cdots, m$. Thus for n large, $m \leq \#F^{-1}(z_n) \leq l$, so $z \in V_l$.

Now let

$$B(z; \delta) = \{\zeta \in \mathbf{C}^n \,|\, |z - \zeta| < \delta\}, \qquad z \in \mathbf{C}^n, \ \delta > 0;$$
$$Z_l = \{z \in W \,|\, \forall \delta > 0 \ \exists\ K \subseteq B(z; \delta) \cap V_l \text{ with } K \text{ measurable and } m_{2n}(K) > 0\}, \qquad l > 0.$$

(It is clear that the set Z_l is closed relative to W. Thus the next assertion shows that either $Z_l = \emptyset$ or $Z_l = W$.)

Assertion 2. *$\forall l > 0$, Z_l is open and $Z_l \subseteq V_l$.*

Proof of Assertion 2. Let $z^0 \in Z_l$. Choose $\delta > 0$ such that $B(z^0; \delta) \subseteq W$, and let K be a closed subset of $B(z^0; \delta) \cap V_l$ such that $m_{2n}(K) > 0$. Assertion 1 implies that $K \cap W_j$ is measurable for each j, so $\exists k$, $1 \leq k \leq l$, such that $m_{2n}(W_k \cap K) > 0$.

Fix $z \in B(z^0; \delta)$. It is not hard to show that $\exists \lambda \in \mathbf{C}^n \setminus \{0\}$ such that if $L = \{z + \zeta\lambda \,|\, \zeta \in \mathbf{C}\}$, then $m_2(L \cap W_k \cap K) > 0$. For simplicity let $z = 0$, $\lambda = (1, 0, \cdots, 0)$. Let $G = (f_2, \cdots, f_n)$ and define:

$$\widetilde{W} = \text{component of } \mathbf{C} \setminus f_1(X \cap V(G)) \text{ which contains } 0;$$
$$\widetilde{K} = \{\zeta \in \mathbf{C} \,|\, (\zeta, 0, \cdots, 0) \in W_k \cap K\}.$$

Then $f_1|_{V(G)} \in A_{V(G)}$ and is k-to-1 over $\widetilde{K}$, so the one-dimensional version of Theorem 2 shows that $\forall \lambda \in \widetilde{W}$, $\#[(f_1)^{-1}(\lambda)] \leq k$. (Compare Assertion 3 in the proof of Theorem 11.2 in [7]; the "k" there is the same as our "k" as may be verified from its definition on p. 65.) Thus $\#F^{-1}(0) \leq k$; so $\forall z \in B(z^0; \delta)$, $\#F^{-1}(z) \leq k$, or $B(z^0; \delta) \subseteq V_k$, so $B(z^0; \delta) \subseteq Z_l$. This concludes the proof of Assertion 2.

We are ready to prove (i). Wlog W' is closed. Since $W' = \bigcup_{l=1}^{\infty} W' \cap W_l$, Assertion 1 implies that each $W' \cap W_l$ is measurable, so there is a positive integer k such that $m_{2n}(W' \cap W_k) > 0$. Then $Z_k \neq \emptyset$, so the connectedness of W and Assertion 2 imply that $Z_k = W$; thus $W \subseteq V_k = \bigcup_{l=1}^{k} W_l$. Note also that if $l < k$, the same argument implies that we must have $m_{2n}(W_l) = 0$.

Assertion 3. *If $z^0 \in W_k$, $\exists \delta > 0$ with $B(z^0; \delta) \subseteq W_k$ such that for each component N of $F^{-1}(B(z^0; \delta))$ we have:*

$$F|_N \colon N \to B(z^0; \delta) \text{ is a homeomorphism;}$$
$$\forall g \in A, \ g \circ (F|_N)^{-1} \text{ is holomorphic on } B(z^0; \delta).$$

Thus it follows that $F : F^{-1}(W_k) \to W_k$ *is a k-sheeted covering map.*

Proof of Assertion 3. Fix $z^0 \in W_k$, and let $F^{-1}(z^0) = \{x_1, \cdots, x_k\}$. Let $N_1, \cdots, N_k$ be disjoint closed neighborhoods of $x_1, \cdots, x_k$. Lemma 3 implies that $F(N_j)$ is a neighborhood of z^0 for each j. Choose $\delta > 0$ such that

$$\overline{B}(z^0; \delta) = \{z \in \mathbf{C}^n \mid |z - z^0| \le \delta\} \subseteq W \cap \left(\bigcap_{j=1}^{k} F(N_j)\right).$$

Since $W = V_k$, we have that for each $z \in \overline{B}(z^0; \delta)$ and for each j, $\exists$ a unique $x \in N_j$ with $F(x) = z$. Let $M_j = N_j \cap F^{-1}(\overline{B}(z^0; \delta))$, $1 \le j \le k$. Since M_j is compact and F maps M_j one-to-one onto $\overline{B}(z^0; \delta)$, the restriction of F to M_j is a homeomorphism onto $\overline{B}(z^0; \delta)$. Let $X_j = \{x \in M_j \mid |F(x) - z^0| = \delta\} = \partial M_j$. Corollary 1 implies that $\partial_{n-1} A_{M_j} \subseteq X_j$. Fix j and let $N = M_j \backslash X_j$. Theorem 1 now shows that $\forall g \in A$, $g \circ (F|_N)^{-1}$ is holomorphic on $\overline{B}(z^0; \delta)$. This completes the proof of Assertion 3.

To prove (ii), define $p_j : W_k \to M$, $1 \le j \le k$, by requiring that $F^{-1}(z) = \{p_1(z), \cdots, p_k(z)\}$, $z \in W_k$, the p_j being otherwise arbitrary. For each $f \in A$, define

$$\Delta[f](z) = \begin{cases} \prod_{i \ne j} (f(p_i(z)) - f(p_j(z))), & z \in W_k, \\ 0, & z \in W \backslash W_k. \end{cases}$$

Assertion 3 shows that $\forall f \in A$, $\Delta[f]$ is holomorphic on the open set W_k, and the remark preceding Assertion 3 shows that W_k is dense in W. If $z^0 \in W \backslash W_k$, then $\# F^{-1}(z^0) < k$ and Lemma 3 implies that for each $f \in A$,

$$\lim_{z \to z^0;\, z \in W_k} \Delta[f](z) = 0.$$

Thus we may apply Rado's theorem (the $n = 1$ case is Theorem 10.6 in [7]) to conclude that $\forall f \in A$, $\Delta[f]$ is holomorphic on W. Since the functions in A separate points on M,

$$\bigcup_{j=1}^{k-1} W_j = W \backslash W_k = \bigcap_{f \in A} \{z \in W \mid \Delta[f](z) = 0\}.$$

Thus $\bigcup_{j=1}^{k-1} W_j$ is a proper analytic subvariety of W.

Now we have almost all the information needed to conclude that $F : F^{-1}(W) \to W$ is a k-sheeted (branched) analytic covering of W in the sense of [5, p. 101]. In fact, the only conditions in this definition of analytic cover which we have not already verified are that the restriction of F to $F^{-1}(W)$ is a proper map and that $F^{-1}(W\setminus\bigcup_{j=1}^{k-1} W_j)$ is dense in $F^{-1}(W)$. The first condition is an immediate consequence of the fact that F is defined and continuous on all of the compact set M, and the second condition follows from an application of Lemma 3 as in Assertion 3.

The fact that an analytic cover is an analytic space may be found in [4, Theorem 32]. If $f \in A$, Assertion 3 and Definition 4 in [5, p. 101] now imply that the restriction of f to $F^{-1}(W)$ is holomorphic.

Theorem 3. *Let $n > 0$ and let $\partial_{n-1}A \subseteq X$. Let $F \in A^n$ and let W be a component of $\mathbf{C}^n\setminus F(X)$. Assume that $\forall z \in W$, $F^{-1}(z)$ is at most countable. Then $\exists$ an open dense subset U of $F^{-1}(W)$ and a sheaf $\mathcal{O}$ of germs of functions on U such that $(U; \mathcal{O})$ is an n-dimensional complex analytic manifold and $\forall f \in A$, f is holomorphic on U.*

Proof. The proof is virtually identical with that of the Theorem in [1], so we shall not repeat it here. (Simply substitute Theorem 1, Lemma 2 and Lemma 3 above for Lemmas 1, 2 and 3 in the proof of the Theorem in [1].)

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