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GENERALIZED FLAG MANIFOLDS BOUND EQUIVARIANTLY

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ABSTRACT. Given a compact, connected lie group G and a maximal torus T, we give a simple, explicit construction of a G-manifold M which bounds the homogeneous space G/T equivariantly.

Let G be a compact, connected lie group with a maximal torus T. We will construct a compact manifold M with a G-action, and a G-equivariant imbedding $G/T \to M$ which identifies G/T with the boundary of M. It is known [1] that the Pontryagin and Stiefel-Whitney *classes* of G/T vanish, so certainly the corresponding characteristic numbers; and hence by general results of cobordism theory, G/T bounds. However, this result about the characteristic classes requires a detailed study of the cohomology of G/T. In any case, the deduction from cobordism theory does not provide any "simple" explicit manifold bounding G/T, let alone equivariantly.

We start with the well-known decomposition for the lie algebra $\mathfrak{g}$ of G, as an oriented $\mathrm{ad}(T)$-module: $\mathfrak{g} = \mathfrak{t} \oplus \Sigma_{\alpha > 0}\, \mathfrak{g}_\alpha$, where $\mathfrak{t}$ is the lie algebra of T and $\mathfrak{g}_\alpha$ are irreducible, oriented $\mathrm{ad}(T)$-planes corresponding to the (positive) roots $\alpha: \mathfrak{t} \to R$. The subspace $\mathfrak{c}_\alpha$ generated by $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ is actually a lie subalgebra [2, Chapter 6][1] isomorphic to $\mathfrak{a} \oplus \mathrm{su}(2)$, where $\mathfrak{a} = \mathrm{Ker}(\alpha)$ is an abelian ideal, and $\mathrm{su}(2)$ is generated by the coroot $H_\alpha \in \mathfrak{t}$ and $\mathfrak{g}_\alpha$. Denoting by $C_\alpha \subseteq G$ the connected subgroup corresponding to $\mathfrak{c}_\alpha$, it is easy to see that $C_\alpha/T \simeq S^2$, the two-sphere. Here is a quick proof: C_α/T is a compact two-manifold, and since any compact lie group modulo its maximal torus is simply-connected, it must be S^2. Note that S^2 acquires a natural orientation from $\mathfrak{g}_\alpha$.

Now consider the homogeneous fibre-bundle $C_\alpha/T \to G/T \to G/C_\alpha$. This exhibits G/T as $G \times C_\alpha(S^2)$ as a G-space, with G acting on the latter space by left multiplication in the first factor. Our main observation is that

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[1] Serre [2] only discusses the semisimple case, but since $\mathfrak{g}$ is reductive, the same argument applies.

the C_α-action on S^2 is equivalent to one through $SO(3)$, hence extends to a C_α-action on the three-disk D^3. This can be seen directly, if one writes out the isomorphism $C_\alpha/T \simeq S^2$ using the lie theory above. For our purposes, we can invoke "uniformization" since C_α preserves some complex structure [1, §12]—or equivalently, some Riemannian metric—on S^2. Therefore we construct the manifold $M = G \times C_\alpha(D^3)$, with G acting on M by left multiplication in the first factor. The obvious inclusion $G \times C_\alpha(S^2) \longrightarrow G \times C_\alpha(D^3)$ then gives the required imbedding.

In a word, we have "filled in" the two-spheres in the fibre-bundle above. Note that there is a G-equivariant fibre-map $\pi: M \longrightarrow G/C_\alpha$ with fibre D^3, and our homogeneous bundle above is the "boundary-bundle" of $D^3 \longrightarrow M \xrightarrow{\pi} G/C_\alpha$.

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