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OMITTING TYPES: APPLICATION TO DESCRIPTIVE SET THEORY

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ABSTRACT. The omitting types theorem of infinitary logic is used to prove that every small Π_1^1 set of analysis or any small Σ_1 set of set theory is constructible.

In what follows we could use either the omitting types theorem for infinitary logic or the same theorem for what Grilliot [2] calls (ϵA)-logic. I find the latter more appealing. Suppose $\mathfrak{L}$ is a finitary logical language containing the symbols of set theory as well as a constant symbol $\bar{a}$ for each a in the transitive set A. For this language we will use only (ϵA)-models, that is to say, end extensions of the model $\langle A, \epsilon\rangle$. Corresponding to this restricted notion of model is a strengthened notion of proof, (ϵA)-logic. In addition to the usual finitary rules of proof, this logic contains rules R_a for each a in A. Rule R_a says "From $\phi(\bar{b})$ for each b in a, you may conclude $\forall x \in \bar{a}\,\phi(x)$." This logic satisfies both the completeness and omitting types theorems. If A is admissible and T is Σ on A, the predicate $T \vdash_{\epsilon A} \phi$ is also Σ on A. Proofs follow easily from the corresponding theorems of infinitary logic.

A Π_1^1 set is *small* if it has no perfect subsets. Using the theorem that every set Σ_1^1 in the parameter α having a member not hyperarithmetic in α has a perfect subset, a number of people[1] have observed that every small Π_1^1 set is contained in the set S defined as follows: $\alpha \in S$ iff α is hyperarithmetic in every β with $\omega_1^\alpha \leq \omega_1^\beta$. Here ω_1^α is the first ordinal not recursive in α. It has also been observed that $S = Q$, where Q is the set of α which are constructible by stage ω_1^α in the constructible hierarchy. Since $Q \subseteq L$, in order to prove that no small Π_1^1 set has a nonconstructible element,[2]

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[1] These include Kechris, Sachs, and Guaspari. Guaspari claims the record of 7 different characterizations of S.

[2] This theorem was first proven in [4] and [5]. The above-quoted strengthening of the theorem to $S=Q$ has previously been proven using essentially the same forcing techniques as the original theorem.

it suffices to show half of this equality, namely $S \subseteq Q$.

Theorem 1. $S \subseteq Q$.

Proof. Let α be an arbitrary set of integers and let σ be ω_1^α. Consider the language for $(\epsilon\sigma)$-logic which contains an additional unary predicate letter F. Let T be the ZF set theory augmented by the axioms "$F \subseteq \omega$" and each instance of "$\overline{\tau} < \omega_1^F$" for $\tau < \sigma$. The type D is $\{x \subseteq \omega\} \cup \{\overline{n} \in x: n \in \alpha\} \cup \{\overline{n} \notin x: n \notin \alpha\}$. Since T has an $(\epsilon\sigma)$-model, either D is principal in $(\epsilon\sigma)$-logic or T has an $(\epsilon\sigma)$-model omitting D.

In the first case let $\phi(x)$ be a generator of D. Then $\alpha = \{n: T \vdash_{\epsilon\sigma} \phi(x) \rightarrow \overline{n} \in x\}$ and $\sim\alpha = \{n: T \vdash_{\epsilon\sigma} \phi(x) \rightarrow \overline{n} \notin x\}$, thus $\alpha \in \Delta$ on the admissible set $L(\sigma)$ and hence is in $L(\sigma)$; thus $\alpha \in Q$.

In the second case there is an $(\epsilon\sigma)$-model for T not containing α. Letting β be the interpretation of F in that model, we see that $\omega_1^\alpha \leq \omega_1^\beta$, but α is not hyperarithmetic in β. Thus $\alpha \notin S$. □

Definition. For x and y hereditarily countable sets, x is *hyperarithmetic* in y if $x \in A$ for every admissible A with $y \in A$. Note that for x and y sets of integers, this definition is equivalent to the other usual ones.

Lemma. *If σ is a countable ordinal and $A \subseteq \sigma$ is not hyperarithmetic in σ, then there is a well ordering of integers $<$ of type σ with A not hyperarithmetic in $<$.*

Proof. We use $(\epsilon\sigma + 1)$ logic. Let $\mathfrak{L}$ be the language for that logic augmented by the binary relation symbol $<$. T is ZF set theory augmented by "$<$ is a well ordering of integers of type σ." The type D is $\{x \subseteq \overline{\sigma}\} \cup \{\overline{\rho} \in x: \rho \in A\} \cup \{\overline{\rho} \notin x: \rho \notin A\}$. As in the previous proof, if D were principal, A would be hyperarithmetic in σ. Thus D is not principal and so T has an $(\epsilon\sigma + 1)$-model not containing A. Since the well-founded part of any model of ZF is admissible, this completes the proof. □

Definition. A set $A \subseteq P(\rho)$ (the power set of ρ) is analytic in σ if there is a formula ϕ of set theory with A defined by the condition: "There is a transitive set a of rank $\leq \sigma$ with $x \in a$ and $\langle a, \epsilon\rangle \models \phi(x)$."

Theorem 2. *If $A \subseteq P\sigma$ is analytic and has an element not hyperarithmetic in σ, then A has $2^{\aleph_0}$ elements.*

Proof. Suppose $x \in A$ is not hyperarithmetic in σ. Let $<$ be a well ordering of integers of type σ, with x not hyperarithmetic in $<$. This ordering obviously induces a simple map from σ 1-1 onto ω, and hence a functional F recursive in $<$ mapping $P\sigma$ 1-1 onto $P\omega$. Clearly $F(x)$ is not hyperarithmetic in $<$, so it remains to show that $\{F(y): y \in A\}$ is Σ^1_1 in $<$. The theorem would then follow directly from the corresponding theorem for Σ^1_1 sets quoted in the second paragraph. $\{F(y): y \in A\}$ can be defined as the set of z satisfying "there is a binary relation R on the integers which can be mapped into $<$ and an integer n such that z is the image under F of the transitive collapse of n and $\langle \omega, R\rangle \models \phi(n)$." This condition can be routinely shown to be Σ^1_1. □

Theorem 3. *Suppose A is Σ on HC (the set of hereditarily countable sets) and has a nonconstructible element, then A has $2^{\aleph_0}$ elements.*

Proof. We may as well assume that A is transitive, since its transitive closure is also Σ on HC, and mod $\aleph_0$ has the same cardinal as A. Let F be the usual Σ isomorphism of L_{ω_1} onto ω_1. For the same reason as above, we may as well assume that for $x \in A$, $\{F(y): y \in x\}$ is also in A. By these two assumptions A contains a nonconstructible set of ordinals. Let σ be a countable ordinal and x_0 be an element of $(A - L) \cap P\sigma$. Let $\rho \geq \sigma$ be such that $V_\rho \models \phi(x_0)$, where ϕ is the Σ definition of A. Then the set of $y \subseteq \sigma$ satisfying "there is a transitive set a with rank $(a) \leq \rho$ and $y \in a$ and $\langle a, \epsilon\rangle \models \phi(y)$" is a subset of A, analytic in ρ, containing the nonconstructible element x_0. Since x_0 is not hyperarithmetic in any ordinal, this, with Theorem 2, completes the proof. □

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