

## DIRECT PRODUCT DECOMPOSITION OF ALTERNATIVE RINGS

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**ABSTRACT.** It is shown that any alternative ring  $A$  equipped with the relation  $\leq$ , defined by  $x \leq y$  if and only if  $xy = x^2$ , is isomorphic to a direct product of alternative division rings if and only if the relation  $\leq$  is a partial order on  $A$  such that  $A$  is hyperatomic and orthogonally complete.

The result stated in the abstract was first proved by Abian [1] for the commutative associative case. Later, Chacron [4] extended this result to an arbitrary associative ring. While Chacron proved this by using the axiom of choice or a subdirect sum representation of any associative ring without non-zero nilpotent elements, Abian [2] recently gave another method to prove the result of Chacron without using the axiom of choice. It is the aim of this paper to extend the above result for the associative case to any alternative ring.

**1. Preliminaries.** An alternative ring is a nonassociative ring  $A$  satisfying  $x^2y = x(xy)$  and  $yx^2 = (yx)x$  for all  $x, y \in A$ . In terms of the associator  $(x, y, z) = (xy)z - x(yz)$ , this is to say  $(x, x, y) = (y, x, x) = 0$ . Thus an associative ring is alternative, and there exist alternative rings without non-zero nilpotent elements which are not associative. For this, see, for example, [3] or [6].

We recall some of the well-known properties in an alternative ring  $A$  on which our proofs will be based. Artin's theorem in  $A$  says that

(1) any subring of  $A$  generated by two elements is associative.

Moufang identities are

$$(2) \quad (axa)y = a[x(ay)],$$

$$(3) \quad a(xy)a = (ax)(ya) \quad \text{for all } x, y \in A.$$

Also, the following generalized version of Artin's theorem is proved in [3]:

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- (4) if  $(a, b, c) = 0$ , the subring of  $A$  generated by  $a, b, c$  is associative.

If  $A$  is an alternative ring without nonzero nilpotent elements, the following, in view of Artin's theorem, can be shown to be exactly the same as the associative case [2].

- (5)  $x^2 = 0$  for  $x \in A$  if and only if  $x = 0$ .  
 (6)  $xy = 0$  for  $x, y \in A$  if and only if  $yx = 0$ .  
 For any elements  $x, y, t \in A$ ,  $xy = xt$   
 (7) if and only if  $yx = tx$ .

Hence, in particular, if  $xy = x^2$  then  $yx = yx$ .

**2. Main section.** In what follows,  $A$  always stands for an alternative ring without nonzero nilpotent elements. First we prove a sequence of lemmas which are essential for the main result.

**Lemma 1.** *Let  $A$  be an alternative ring. The relation defined by  $x \leq y$  if and only if  $xy = x^2$  is a partial order on  $A$  if and only if  $A$  has no nonzero nilpotent elements.*

**Proof.** Suppose  $A$  has no nonzero nilpotent elements. Then clearly  $\leq$  is reflexive. If  $x \leq y$  and  $y \leq x$ , that is,  $xy = x^2$  and  $yx = y^2$ , then we get  $(x - y)^2 = 0$  and so  $x = y$ . To show transitivity of  $\leq$ , let  $x \leq y$  and  $y \leq z$ . We first show

$$(8) \quad x^2zx = xzx^2 = x^4.$$

Since  $xy = x^2$  and  $yz = y^2$ , we have

$$\begin{aligned} x^2zx &= x^2(zx) = (xy)(zx) = x(yz)x \quad (\text{by (3)}) \\ &= xy^2x = (xy)^2 \quad (\text{by (7)}) \\ &= x^4. \end{aligned}$$

Similarly, we show  $xzx^2 = x^4$ . Using (8) we compute

$$(xzx - x^3)^2 = xzx^2zx - xzx^4 - x^4zx + x^6 = x^4zx - x^6 - x^4zx + x^6 = 0$$

and so

$$(9) \quad xzx = x^3.$$

Finally, from (8) and (9), we derive  $(xz - x^2)^2 = x^3z - x^4 - x^3z + x^4 = 0$ , and so  $xz = x^2$  or  $x \leq z$ . Thus  $\leq$  is transitive and is a partial order on  $A$ .

The converse is easy and the same as in the associative case.

Henceforth any order in  $A$  will mean the relation defined by  $x \leq y$  if and only if  $xy = x^2$ . The following lemma is proved by Hentzel [5]. For convenience, we duplicate it here.

**Lemma 2.** *Let  $x, y, z$  be any elements of  $A$ . Then  $(xy)z = 0$  if and only if  $x(yz) = 0$ .*

**Proof.** By repeated use of (1) and (3), if  $(xy)z = 0$  then

$$\begin{aligned} (x \cdot yz)^3 &= x[yz \cdot (x \cdot yz \cdot x) \cdot yz] = x[yz \cdot (xy \cdot zx) \cdot yz] \\ &= x[(yz \cdot xy)(zx \cdot yz)] = x [(yz \cdot xy)(z \cdot xy \cdot z)] = 0 \end{aligned}$$

and so  $x(yz) = 0$ . The converse is similar.

**Lemma 3.** (i) *For every  $x, a \in A$ , if  $x^2a = 0$  then  $xa = 0$ .*

(ii) *If  $x \leq y$  for  $x, y \in A$ , then  $(x, y, A) = 0$ . Hence for every element  $u \in A$ , the subring of  $A$  generated by  $x, y, u$  is associative.*

(iii) *If  $ab \leq c$  for  $a, b, c \in A$ , then the subring generated by  $a, b, c$  is associative.*

**Proof.** (i) In view of (1), the proof is the same as in the associative case.

(ii) Since  $xy = x^2$ ,

$$\begin{aligned} 0 &= (xy - x^2)u = [x(y - x)]u = x[(y - x)u] \quad (\text{Lemma 2}) \\ &= x(yu - xu) = x(yu) - x(xu) = x(yu) - x^2u. \end{aligned}$$

Hence  $(xy)u = x^2u = x(yu)$  and  $(x, y, u) = 0$  for all  $u \in A$ , as desired.

(iii) In view of (4), it suffices to show  $(a, b, c) = 0$ . Since  $(ab)c = (ab)^2$ , using Lemma 2 we have

$$0 = (ab)(c - ab) = a \cdot b(c - ab) = a(bc - bab) = a(bc) - (ab)^2,$$

and so  $(a, b, c) = 0$ .

**Lemma 4.** *For every element  $x, y, u, v \in A$ ,  $x \leq y$  and  $u \leq v$  imply  $xu \leq yv$ .*

**Proof.** We first show that for every  $v \in A$ ,  $x \leq y$  implies  $xv \leq yv$  and  $vx \leq vy$ . Since  $xy = xx$ , by Lemma 3(ii),  $(xy)v = x(yv) = x(xv)$ . So by (7) we get  $(yv)x = xv$  and  $yvxv = xv$ , so  $xv \leq yv$ ; the other case is similar. Since  $xu \leq yu$  and  $yu \leq yv$ , by transitivity (Lemma 1) we have that  $xu \leq yv$ .

**Lemma 5.** *Let  $e$  be an idempotent in  $A$ . Then:*

- (i)  $ex = xe \leq x$  for every  $x \in A$ .
- (ii)  $e$  is central in  $A$ ; that is,  $e$  commutes with every element in  $A$  and  $(e, A, A) = 0$ .

**Proof.** (i) is trivial. For (ii), one easily checks  $(ex - exe)^2 = (xe - exe)^2 = 0$  and so  $ex = xe$ . Now, for every element  $x, y \in A$ ,

$$\begin{aligned} 0 &= [e(ex - x)](ey) \quad (\text{Lemma 2}) \\ &= e[(ex)(ye) - x(ey)] \quad (\text{Moufang}) \\ &= e[e(xy)e - x(ey)] = e(xy) - e[x(ey)] \quad (\text{Moufang}) \\ &= e(xy) - (exe)y = e(xy) - (ex)y, \end{aligned}$$

as desired.

**Lemma 6.** *Let  $a, s$  be elements in  $A$  such that  $a^2s = a$ . Then we have*

- (i)  $asa = a$ ;
- (ii)  $as$  and  $sa$  are idempotents and  $as = sa$ ;
- (iii) if  $x \leq as$  for  $x \in A$ , then  $x$  is an idempotent;
- (iv)  $(a^2, s, A) = 0$ .

**Proof.** (i) One easily checks  $(asa - a)^2 = 0$ .

(ii) Using (i), we get  $(sa)^2 = (sa)(sa) = s(asa) = sa$ , and similarly  $(as)^2 = as$ . This then implies  $(sa - as)^2 = 0$  and so  $sa = as$ .

(iii) If  $x \leq as$ , then  $x^2 \leq x(as)$  and  $x(as) \leq x$  by Lemma 5(i) since  $as$  is an idempotent. Hence  $x^2 \leq x$  by Lemma 1; that is,  $x^4 = x^3$ . Using this, we get  $(x^3 - x^2)^2 = 0$  and so  $x^3 = x^2$ , which means  $x \leq x^2$  and so, by Lemma 1,  $x^2 = x$ .

(iv) Let  $y$  be any element in  $A$ . From (2) and (6), we get  $0 = (a, as, y) = (a, s, y)a = a(a, s, y)$ . Also, a known identity for  $A$  says that  $(a^2, s, y) = a(a, s, y) + (a, s, y)a$  (see [6, p. 129]). Hence  $(a^2, s, y) = 0$ , as desired.

Following Abian [1], we have

**Definition.** A nonzero element  $a$  in  $A$  is called a hyperatom in  $A$  if and only if, for every element  $x$  in  $A$ ,

- (i)  $x \leq a$  implies  $x = 0$  or  $x = a$ , and
- (ii)  $ax \neq 0$  implies  $a(xs) = a$  for some element  $s$  in  $A$ .

We now prove the following crucial lemma.

**Lemma 7.** *Let  $x$  be a nonzero element in  $A$ . If  $a \leq x$  for some nonzero hyperatom  $a$  in  $A$ , then there exists an idempotent hyperatom  $e$  such that  $ex \neq 0$ .*

**Proof.** Since  $a$  is a hyperatom, by definition there exists an element  $s \in A$  with  $a^2s = a$ . Then by Lemma 6(ii),  $as = sa$  is an idempotent. Hence it suffices to show  $as$  is a hyperatom and  $(as)x \neq 0$ . Let  $y \leq as$  for  $y \in A$ . Then  $y^2 = y(as)$  and  $ya \leq asa = a$  by Lemmas 4 and 6(i). Thus  $ya = 0$  or  $ya = a$ . If  $ya = 0$ ,  $0 = (ya)s = y(as) = y^2$  by Lemma 2 and so  $y = 0$ . Suppose now  $ya = a$ . Then we get  $(a, s, y) = 0$  since  $y$  is an idempotent by Lemma 6(iii). Thus  $y = y^2 = y(as) = (ya)s = as$ . Hence  $y = 0$  or  $y = as$ . Now suppose  $(as)y \neq 0$  for  $y \in A$ . Then  $a(sy) \neq 0$  by Lemma 2 and so  $a^2(sy) \neq 0$ , but then  $a^2(sy) = (a^2s)y = ay \neq 0$  by Lemma 6(iv). Since  $a$  is a hyperatom, there exists an  $r \in A$  such that  $a(yr) = a$ , and hence  $(a^2s)(yr) = asa = a^2ssa$ ;

$$0 = (a^2s)(yr - sa) = a^2[s(yr - sa)] \quad (\text{Lemma 2}),$$

and so

$$0 = a[s(yr - sa)] = (as)(yr - sa).$$

Hence  $(as)(yr) = (as)^2 = as$ , and this proves  $as$  is a hyperatom. Finally if we let  $e = as$ , then

$$\begin{aligned} ex &= (sa)x = s(ax) \quad (\text{by Lemma 3(ii)}) \\ &= sa^2 = (sa)a = asa = a \neq 0 \end{aligned}$$

by Lemmas 5(ii) and 6(i).

**Lemma 8.** *The set  $E = \{e_i\}_{i \in I}$  of all idempotent hyperatoms in  $A$  is an orthogonal set in  $A$ , and each  $e_iA$  is an alternative division ring such that  $e_iA \cap e_jA = 0$  if  $i \neq j$ .*

Since each  $e_iA$  has no zero divisors, noting that each  $e_i$  is central and that if every nonzero element in an alternative ring has a right inverse it has also a left inverse [6, p. 131], the proof is the same as in the associative case.

As in the associative case, we make the following

**Definition.** Let  $A$  be an alternative ring.

(i)  $A$  is called hyperatomic if for every nonzero element  $r$  in  $A$  there exists a hyperatom  $a$  in  $A$  such that  $a \leq r$ .

(ii)  $A$  is called orthogonally complete if  $\sup S$  exists for every orthogonal subset  $S$  of  $A$ .

The following lemma is an easy consequence of Lemma 7 (also see [1] and [4]).

**Lemma 9.** *Let  $A$  be hyperatomic and  $E = \{e_i\}_{i \in I}$  be the set of idempotent hyperatoms in  $A$ . Then we have*

- (i) for every  $a \in A$ ,  $\sup(e_i a)$  exists and  $a = \sup(e_i a)$ ;  
(ii) the function  $f$  defined by  $f(a) = (e_i a)_{i \in I}$  is a monomorphism from  $A$  into a direct product of the alternative division rings  $e_i A$ .

For the proof of the main result, we need one more lemma.

**Lemma 10.** Let  $\{x_i\}_{i \in I}$  be a subset of the alternative ring  $A$  such that  $\sup_i x_i$  exists. Then for every element  $a \in A$ ,  $\sup_i(ax_i)$  exists and furthermore,

$$a \sup_i x_i = \sup_i(ax_i).$$

**Proof.** Let  $u$  be any upper bound of  $\{ax_i\}_{i \in I}$ . We proceed as in [2]. Since  $ax_i \leq u$ ,

$$(10) \quad u(ax_i) = (ax_i)^2 \quad \text{for every } i \in I.$$

By Lemma 4,  $a \sup_i x_i$  is an upper bound of  $\{ax_i\}_{i \in I}$  and so

$$(11) \quad \left(a \sup_i x_i\right)(ax_i) = (ax_i)^2 \quad \text{for every } i \in I.$$

From (10) and (11) we have

$$u(ax_i) = \left(a \sup_i x_i\right)(ax_i) \quad \text{for every } i \in I.$$

Thus

$$x_i^2 = \left(\sup_i x_i\right)x_i + u(ax_i) - \left(a \sup_i x_i\right)(ax_i).$$

Since  $ax_i \leq u$  for every  $i \in I$ , from Lemma 3(iii) and 3(ii) applied to  $x_i$ ,  $\sup_i x_i$ ,  $a$ , we have

$$\begin{aligned} x_i^2 &= \left(\sup_i x_i\right)x_i + (ua)x_i - \left[\left(a \sup_i x_i\right)a\right]x_i \\ &= \left[\sup_i x_i + ua - \left(a \sup_i x_i\right)a\right]x_i = x_i \left[\sup_i x_i + ua - \left(a \sup_i x_i\right)a\right]. \end{aligned}$$

Thus

$$x_i \leq \sup_i x_i + ua - \left(a \sup_i x_i\right)a$$

for every  $i \in I$ , which means

$$\left(\sup_i x_i\right)^2 = \left(\sup_i x_i\right) \left[\sup_i x_i + ua - \left(a \sup_i x_i\right)a\right],$$

which implies

$$(12) \quad \left(\sup_i x_i\right)(ua) = \left(\sup_i x_i\right)\left(a \sup_i x_i\right)a.$$

Setting  $x = \sup_i x_i$ , from (12) we get  $x(ua) = xaxa$ . Applying Lemma 2 to this,

$$0 = x(ua - axa) = x \cdot (u - ax)a = x(u - ax) \cdot a = (xu)a - xaxa.$$

Hence  $(xu)a = x(ua) = (xa)^2$  and  $a(xu) = (ax)u = (ax)^2$  or  $ax \leq u$ . Since  $ax$  is an upper bound of  $\{ax_i\}_{i \in I}$ , this proves  $ax = a \sup_i x_i = \sup_i (ax_i)$ , as desired.

**Remark.** Lemmas 1, 4, and 10 generalize the results of Abian [2] to the alternative case. Also, our present results are extended to right alternative rings of characteristic  $\neq 2$ , since Kleinfeld [7] shows that right alternative rings of characteristic  $\neq 2$  without nonzero nilpotent elements are alternative.

We are now prepared to prove the main result.

**Theorem.** Any alternative ring  $A$  equipped with the relation  $\leq$ , defined by  $x \leq y$  if and only if  $xy = x^2$ , is isomorphic to a direct product of alternative division rings if and only if the relation  $\leq$  is a partial order on  $A$  such that  $A$  is hyperatomic and orthogonally complete.

The proof of the Theorem is a consequence of Lemmas 9, 8, and 10, and thus the same as in the associative case (see [1] and [4]).

The following example shows that Lemma 1 and the Theorem do not hold for Jordan rings.

**Example.** Let  $Q$  be the real quaternions with the standard basis  $1, i, j, k$ . Then  $Q$  under the product  $a \cdot b = \frac{1}{2}(ab + ba)$  becomes a Jordan ring  $Q^+$  without nonzero nilpotent elements. One easily checks in  $Q^+$  that  $i + j \leq 2i$  and  $2i \leq 2i + j$  hold but  $i + j \leq 2i + j$  does not. Thus the relation  $\leq$  is not a partial order on  $Q^+$ . Also,  $Q^+$  is a Jordan division ring in the sense that  $U_a = 2R_a^2 - R_a^2$  is invertible on  $Q^+$  for every  $a \neq 0$  in  $Q^+$ , where  $R_a$  is the right multiplication in  $Q^+$  by  $a$ .

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