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A CHARACTERIZATION OF STRICTLY CONVEX BANACH SPACES

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ABSTRACT. A very short proof is given for Petryshyn's characterization of strictly convex Banach spaces [2].

Let X be a real Banach space. Now X is strictly convex (s.c.) iff every $f \in X^*$ attains a maximum on at most one point of the unit sphere [1]. A *duality mapping* is a function $J: X \to 2^{X^*}$ which satisfies $J(x) = \{f \in X^*: f(x) = \|f\| \|x\| \text{ and } \|f\| = \|x\|\}$. We say that J is *strictly monotone* if for every $x \neq y$ and every $f \in J(x)$, $g \in J(y)$ we have $(f - g)(x - y) > 0$ [2]. For $f \in J(x)$, $g \in J(y)$, we see from the following expansion due to Browder that

$$(f - g)(x - y) = (\|x\| - \|y\|)^2 + (\|f\| \|y\| - f(y)) + (\|g\| \|x\| - g(x)) \geq 0.$$

Theorem (Petryshyn). *X is s.c. iff J is strictly monotone.*

Proof. Since each term of Browder's expansion is nonnegative we have: J is not strictly monotone $\Leftrightarrow \exists\, x \neq y$, $f \in J(x)$, $g \in J(y)$ with $(f - g)(x - y) = 0 \Leftrightarrow \exists\, f \in X^*$ with

$$f(x/\|x\|) = f(y/\|y\|) = \|f\| = \|x\| = \|y\|$$

$\Leftrightarrow \exists\, f \in X^*$ that attains a maximum at two points of the unit sphere.

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