PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 1, January 1975

THE DUAL OF A THEOREM OF BISHOP AND PHELPS

GEORGE LUNA

ABSTRACT. We dualize a theorem of Bishop and Phelps by showing that in the dual of a Banach space the intersection of a weak* closed finite codimensional linear variety and a weak* closed convex subset C contains a norm dense set of weak* support points of C. We use this theorem to obtain a result which is related to an abstract approximation problem of Deutsch and Morris.

If C is a weak* closed convex subset of E^*, the dual of a Banach space E, then by a weak* support point of C we mean a point $z^* \in C$ for which there exists $z \in E\backslash\{0\}$ such that $S_C(z) = \langle z, z^* \rangle$. ($S_C$ is the *support function* for C and is defined for each $x \in E$ by $S_C(x) = \sup\{\langle x, x^*\rangle \mid x^* \in C\}$.) In [5], Phelps showed that the set of weak* support points of C is large in the sense of the following theorem.

Theorem 1 [Phelps]. *If E is a Banach space and C is a weak* closed convex subset of E^*, then the weak* support points of C are norm dense in the norm boundary of C.*

As a consequence of Theorem 1 and the following lemma of Bishop-Phelps [1, Lemma 4], we will obtain a dual to [1, Theorem 4]. We remark that although [1, Lemma 4] is stated only for Banach spaces, its proof is valid in any topological vector space.

Lemma 2 [Bishop-Phelps]. *Suppose M is a closed subspace of finite codimension in a topological vector space X, and that C is a convex subset of X. Suppose x_0 is a support point of $C \cap M$ in the subspace M. Then x_0 is a support point of C.*

By the polar C° of a set $C \subseteq E$, we mean the set $\{x^* \in E^* \mid S_C(x^*) \leq 1\}$. If N is a subspace of E, then $N^\perp$ denotes the annihilator of N in E^*, i.e. $N^\perp = \{n^* \in E^* \mid \langle n, n^* \rangle = 0 \text{ for each } n \in N\}$.

Received by the editors November 12, 1973.

AMS (MOS) subject classifications (1970). Primary 46B99; Secondary 41A65, 41A05.

Key words and phrases. Bishop-Phelps, weak* support point, abstract approximation, approximation and interpolation, norm preserving approximation.

Theorem 3. *Let C be a closed convex subset of the Banach space E, and let N be a finite-dimensional subspace of E. Suppose $z^* \in N^\perp \cap$ bdry C°. Then for each $\epsilon > 0$ there exists a weak* support point w^* of C° such that $w^* \in N^\perp$ and $\|w^* - z^*\| \le \epsilon$.*

Proof. We identify the Banach spaces E/N and $N^\perp$. Recall that $N^\perp$ with the relative $\sigma(E^*, E)$ topology is topologically isomorphic with $(E/N)^*$ with the $\sigma((E/N)^*, E/N)$ topology. Thus the set $C^\circ \cap N^\perp$ is a $\sigma(N^\perp, E/N)$ closed convex subset of $N^\perp$. If $z^* \in$ norm bdry $(C^\circ \cap N^\perp)$ in $N^\perp$, then by Theorem 1 applied to $N^\perp$, there exists a $\sigma(N^\perp, E/N)$ support point w^* of $C^\circ \cap N^\perp$ in $N^\perp$ such that $\|w^* - z^*\| \le \epsilon$. By Lemma 2 the element w^* is a $\sigma(E^*, E)$ support point of C° in E^*.

If $z^* \notin$ norm bdry $(C^\circ \cap N^\perp)$ in $N^\perp$, then we show that z^* is itself a weak* support point of C°. Since $z^* \in$ bdy C°, there exists an element $y^* \in E^* \backslash C^\circ$ such that the segment $(z^*, y^*] \subseteq E^*/C^\circ$. (Otherwise, z^* is an element of the core of C°, and since C° is closed and E^* is of the second category in itself, the core of C° is equal to the interior of C°.) Let $M = \operatorname{span}(N^\perp \cup \{y^*\})$ and note that $N^\perp$ is a hyperplane in M; if we show that $N^\perp$ supports $C^\circ \cap M$ at z^*, then from Lemma 2, we can conclude that the point z^* is a weak* support point of C°. It suffices to show that C° is disjoint from the open half space $\{n^* + ry^* | n^* \in N^\perp \text{ and } r > 0\}$ in M defined by $N^\perp$. If $n^* + ry^* \in C^\circ$ where $r > 0$, then (since z^* is in the $N^\perp$-interior of $C^\circ \cap N^\perp$) there exists $w^* \in C^\circ$ and $\lambda \in (0, 1)$ such that $z^* = \lambda n^* + (1 - \lambda) w^*$. Thus the triangle with vertices $n^* + ry^*$, n^*, and w^* is in C°, and this clearly contradicts the fact that $(z^*, y^*] \subseteq E^*/C^\circ$. This completes the proof.

Let C be a closed convex subset of the Banach space E, and denote by $P(C)$ the set of support functionals of C.

We are going to prove that the intersection of a weak* closed flat of finite codimension with the polar of a closed convex bounded subset C of a Banach space contains a norm dense set of support functionals of C. This will then be shown to be related to an abstract approximation problem of Deutsch and Morris [2].

Proposition 4. *Let E be a Banach space and N a finite-dimensional subspace of E. Suppose $C \subseteq E$ is closed convex and bounded and $0 \in C$. If $x^* \in E^*$ and $S_C(x^*) = 1$, then for each $\epsilon > 0$ there exists $z^* \in P(C) \cap B(x^*; \epsilon) \cap (x^* + N^\perp)$ such that $S_C(z^*) = 1$.*

Proof. Since $S_C(x^*) = 1$, we have $x^* \in$ bdry C°, hence $0 \in (\text{bdry}(C^\circ - x^*)) \cap N^\perp$. According to Theorem 3 there exists a weak* support point w^* of $C^\circ - x^*$ such that $\|w^*\| \le \epsilon$ and $w^* \in N^\perp$. Let $z^* = w^* + x^*$; it only remains to show that $z^* \in P(C)$ and $S_C(z^*) = 1$. Since z^* is clearly a weak* support point of C°, there exists $z \in E\setminus\{0\}$ satisfying $S_{C^\circ}(z) = \langle z, z^* \rangle$. Because $z \neq 0$ there exists $n^* \in E^*$ such that $\langle z, n^* \rangle > 0$. Since C is bounded, we know C° is radial at 0; hence there exists $\lambda > 0$ such that $\lambda n^* \in C^\circ$ so

$$0 < \langle z, \lambda n^* \rangle \leq S_{C^\circ}(z).$$

Without loss of generality we can suppose

$$S_{C^\circ}(z) = \langle z, z^* \rangle = 1.$$

Thus $z \in C$ (by the bipolar theorem) and we have

$$S_C(z^*) = \langle z, z^* \rangle = 1 \text{ since } z^* \in C^\circ.$$

This completes the proof.

The preceding proposition is related to an abstract approximation problem of Deutsch and Morris [2] called "property (SAIN)" for "simultaneous approximation and interpolation which is norm preserving." In the present context this property (which we call "property (S)") is the following:

If E is a Banach space, M is a dense subset of E^*, and N is a finite-dimensional subspace of E^{**}, then the triple (E^*, M, N) has *property* (S) if for each $\epsilon > 0$ and $x^* \in E^*$, there exists $z^* \in M$ satisfying

$$\|z^* - x^*\| < \epsilon, \quad \|z^*\| = \|x^*\|, \text{ and } z^* - x^* \in N^\perp.$$

Deutsch and Morris established in [2, Theorem 2.3] that in case M is a linear subspace of E^*, then (E^*, M, N) has property (S) only if each element of N either attains its norm at points of M or not at all. This raises the question of what happens if M is the norm dense subset $P(B)$ of E^* (B is the unit ball of E)? Since $P(B)$ is not in general convex (a standing hypothesis on the set M in previous theorems about property (S)), the techniques of [2] do not apply. However, as a corollary to Proposition 4, we obtain the following answer to the question raised above.

Corollary 5. *Let E be a Banach space and B the unit ball of E. The triple $(E^*, P(B), N)$ has property* (S) *for each finite-dimensional subspace $N \subseteq E$.*

We remark that in [4] Lambert showed that in the case where $E = c_0$ and B is the unit ball of c_0, that the triple $(l_1, P(B), N)$ has property (S) for each finite-dimensional subspace N of l_∞. That this result does not hold for general Banach spaces E and finite-dimensional subspaces N of E^{**} is shown by the following example.

Example. There exists a Banach space E and a one-dimensional subspace N of E^{**} such that the triple $(E^*, P(B), N)$ does not have property (S).

Let $E = c_0$ with an equivalent norm such that $E^* = l_1$ is strictly convex; let $x^* \in S(l_1) \setminus P(c_0)$, and choose $x^{**} \in S(E^{**})$ so that $\langle x^{**}, x^* \rangle = 1$, and let $N = Rx^{**}$. Then

$$(x^{**} + (x^{**})^{-1}(0)) \cap B^* = \{x^*\}$$

since $S(l_1)$ is strictly convex; thus

$$S(E^*) \cap P(B) \cap (x^* + (x^{**})^{-1}(0)) = \varnothing.$$

This work is taken in part from the author's thesis done in partial fulfillment of the requirements for the Ph. D. degree at the University of Washington. The author is deeply indebted to Professor R. R. Phelps who supervised this study.

BIBLIOGRAPHY

1. Errett Bishop and R. R. Phelps, *The support functionals of a convex set*, Proc. Sympos. Pure. Math., vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 27–35. MR 27 #4051.

2. Frank Deutsch and Peter D. Morris, *On simultaneous approximation and interpolation which preserves the norm*, J. Approximation Theory 2 (1969), 355–373. MR 40 #6146.

3. Richard Holmes and Joseph Lambert, *A geometrical approach to property* (SAIN), J. Approximation Theory 7 (1973), 132–142.

4. Joseph M. Lambert, *Simultaneous approximation and interpolation in* l_1, Proc. Amer. Math. Soc. 32 (1972), 150–152. MR 45 #797.

5. R. R. Phelps, *Weak* support points of convex sets in* E^*, Israel J. Math. 2 (1964), 177–182. MR 31 #3832.

6. ———, *Some topological properties of support points of convex sets*, Israel J. Math. 13 (1972), 327–336.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY, POMONA, CALIFORNIA 91766

Current address: Department of Mathematics, University of California at San Diego, La Jolla, California 92037