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ON COMPLEX STRICT AND UNIFORM CONVEXITY

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ABSTRACT. Strict and uniform c-convexity of complex normed spaces are introduced as a natural generalization of strict and uniform convexity. It is proved that the complex space $L_1(S, \sigma, \mu)$ is uniformly c-convex. An application to analytic functions is given.

Throughout, the open unit disc in the complex plane is denoted by Δ.

Definition 1. A complex normed space X is called strictly c-convex if $x, y \in X$, $\|x\| = 1$ and $\|x + \zeta y\| \leq 1$ $(\zeta \in \Delta)$ implies $y = 0$.

Definition 2. A complex normed space X is called uniformly c-convex if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\|x + \zeta y\| \leq 1$ $(\zeta \in \Delta)$ and $\|y\| > \epsilon$ implies $\|x\| < 1 - \delta$.

Remark. Clearly uniform c-convexity implies strict c-convexity. It is also clear that we get equivalent definitions by replacing Δ by the set $\{1, -1, i, -i\}$.

Given $\delta \geq 0$, let $\omega_c(\delta)$ be the supremum of $\|y\|$ taken over all $x, y \in X$ such that $\|x\| = 1$ and $\|x + \zeta y\| \leq 1 + \delta$ $(\zeta \in \Delta)$. Then one can prove easily that X is strictly (resp. uniformly) c-convex if and only if $\omega_c(0) = 0$ (resp. $\lim_{\delta \searrow 0} \omega_c(\delta) = 0$).

It is obvious that strict (resp. uniform) convexity of a complex normed space implies its strict (resp. uniform) c-convexity. On the other hand we show that there exist uniformly c-convex spaces with no extreme points on the unit sphere.

Thorp and Whitley (see [6]) have proved that the complex space $L_1(S, \sigma, \mu)$ is strictly c-convex. Here we generalize this by proving that $L_1(S, \sigma, \mu)$ is uniformly c-convex. Namely, we prove

Theorem 1. *Let X be the complex space $L_1(S, \sigma, \mu)$. Let $\delta > 0$ and let $x, y \in X$ satisfy $\|x\| = 1$, $\|x \pm y\| \leq 1 + \delta$, $\|x \pm iy\| \leq 1 + \delta$. Then*

$$\|y\| \leq \delta^{1/2}(4 + 2(1 + 2\delta)^{1/2}). \tag{1}$$

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Proof. With no loss of generality we may and do assume that $x(s)$ is finite everywhere on S. Denote $P = \{s \in S \mid y(s) \neq 0\}$ and define $h(s) = x(s)/y(s)$ $(s \in P)$. Let $c > 0$ be arbitrary. Define $P_1 = \{s \in P: |h(s)| > c\}$, $P_2 = \{s \in P: |h(s)| \leq c\}$. Clearly P_1, P_2 are measurable. Since $\|x\| = 1$ we have

$$\int_{P_1} |y(s)| = \int_{P_1} \frac{|x(s)|}{|h(s)|} \leq \frac{1}{c} \int_{P_1} |x(s)| \leq \frac{1}{c}. \tag{2}$$

Further, our assumptions imply that

$$\int_S |x(s) + y(s)| + |x(s) - y(s)| + |x(s) + iy(s)| + |x(s) - iy(s)| - 4|x(s)| \leq 4\delta$$

(note that the integrand is nonnegative by the triangle inequality) so defining

$$F(s) = |h(s) + 1| + |h(s) - 1| + |h(s) + i| + |h(s) - i| - 4|h(s)| \quad (s \in P)$$

we have $\int_P F(s)|y(s)| \leq 4\delta$.

Since F is nonnegative by the triangle inequality it follows that

$$\int_{P_2} F(s)|y(s)| \leq 4\delta. \tag{3}$$

Now denote $f(r, \theta) = |re^{i\theta} + i| + |re^{i\theta} - i| - 2r$, $r \geq 0$, θ real. Then for fixed r the function $\theta \mapsto f(r, \theta)$ is continuous, nonnegative, increasing on $(-\pi/2, 0)$, decreasing on $(0, \pi/2)$ and satisfying $f(r, -\theta) = f(r, \theta)$, $f(r, \theta + \pi) = f(r, \theta)$ for all θ. Hence

$$f(r, \theta) + f(r, \theta + \pi/2) \geq f(r, \pi/4) \quad \text{for all } \theta.$$

Since

$$f(r, \pi/4) = (r^2 + 2^{1/2} r + 1)^{1/2} + (r^2 - 2^{1/2} r + 1)^{1/2} - 2r,$$

it follows that $(f(r, \pi/4) + 2r)^2 \geq 4r^2 + 2$, so $f(r, \pi/4) \geq (4r^2 + 2)^{1/2} - 2r$, where the term on the right is positive and decreasing in r. Since

$$F(s) = f(|h(s)|, \arg h(s)) + f(|h(s)|, \arg h(s) + \pi/2) \quad (s \in P)$$

and since $|h(s)| \leq c$ $(s \in P_2)$ it follows that

$$|F(s)| \geq (4|h(s)|^2 + 2)^{1/2} - 2|h(s)| \geq (4c^2 + 2)^{1/2} - 2c \quad (s \in P_2).$$

Now (3) implies

$$\int_{P_2} |y(s)| \leq 2\delta((4c^2 + 2)^{1/2} + 2c)$$

which, together with (2), gives

$$\|y\| = \int_S |y(s)| = \int_{P_1} + \int_{P_2} |y(s)| \leq 1/c + 2\delta((4c^2 + 2)^{1/2} + 2c),$$

and (1) follows by taking $c = 1/(2\delta^{1/2})$. Q.E.D.

In particular, the complex space $L(0, 1)$ is uniformly c-convex but its unit sphere has no extreme points.

Applications. Let f be a (weakly) analytic function from Δ to a complex normed space X satisfying $\|f(\zeta)\| \leq 1$ $(\zeta \in \Delta)$. Thorp and Whitley (see [6]) have shown that the implication $\|f(0)\| = 1 \Rightarrow f$ is a constant is true in general (which is called the strong maximum modulus theorem) if and only if X is strictly c-convex. Here we prove that given $\zeta \in \Delta - \{0\}$, the implication $\|f(0)\|$ is close to $1 \Rightarrow f(\zeta)$ is close to $f(0)$ is true in general if and only if X is uniformly c-convex. The only if part follows from the definition of uniform c-convexity while the if part follows from

Theorem 2. *Let X be a complex normed space and let $f: \Delta \to X$ be a (weakly) analytic function, satisfying $\|f(\zeta)\| \leq 1$ $(\zeta \in \Delta)$. Then*

$$\|f(\zeta) - f(0)\| \leq (2|\zeta|/(1 - |\zeta|))\omega_c(1 - \|f(0)\|) \quad (\zeta \in \Delta). \tag{4}$$

Proof. If u is a bounded linear functional on X of norm 1 then $|u(f(\zeta))| \leq 1$ $(\zeta \in \Delta)$. Since $\zeta \mapsto u(f(\zeta))$ is analytic on Δ it follows by the lemma of Harris (see [1, p. 1015]) that

$$|u(f(0))| + (1 - |\zeta|)|u(f(\zeta)) - u(f(0))|/2|\zeta| \leq 1 \quad (\zeta \in \Delta - \{0\}).$$

Let $\delta = 1 - \|f(0)\|$ and let w be a unit vector in X with $\|w - f(0)\| = \delta$. Then

$$\|w + \lambda(1 - |\zeta|)(f(\zeta) - f(0))/2|\zeta|\| \leq 1 + \delta \quad (\zeta \in \Delta - \{0\}, \lambda \in \Delta)$$

which, by definition of ω_c implies (4). Q.E.D.

If X is uniformly c-convex then $\lim_{\delta \searrow 0} \omega_c(\delta) = 0$ so (4) implies that $\|f(0)\|$ being close to 1, $f(\zeta)$ is close to $f(0)$, uniformly on closed subdiscs. Note that Theorem 2 still holds if Δ is replaced by the open unit ball of arbitrary complex normed space and the weak analyticity by the weak G-analyticity, respectively.

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