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DERIVATIONS OF C^*-ALGEBRAS WHICH ARE NOT DETERMINED BY MULTIPLIERS IN ANY QUOTIENT ALGEBRA

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ABSTRACT. An example is given to show that, in a C^*-algebra without unit, a derivation is not necessarily determined by a multiplier in some quotient algebra.

When one deals with the derivations of a C^*-algebra without unit, a reasonable replacement for an inner derivation is the derivation determined by a multiplier. By the multiplier algebra $M(A)$ of a C^*-algebra A, we mean the idealizer of A in the universal enveloping von Neumann algebra A'' (i.e. the second dual of A). It happens, however, that the definition does not depend on the von Neumann algebra which contains A; that is, the idealizer in that von Neumann algebra is isomorphic to $M(A)$ by an isomorphism which fixes all elements of A [1, Proposition 2.4]. Let δ be a derivation of A. We say that δ is determined by a multiplier if it becomes inner when it is extended to the derivation of $M(A)$, i.e. if we can choose a multiplier of A as a generator of δ. In general, a derivation may not be determined by a multiplier. However, one may ask whether the restriction of δ to some small ideal is determined by a multiplier. An example for which the answer to this question is negative is shown in [2, 6.5]. Now there arises another basic question for δ: whether we can find an ideal I such that the induced derivation of δ in the quotient algebra A/I is determined by a multiplier. When the algebra has a unit this is true by Sakai's theorem on derivations of simple C^*-algebras [4]. However, we show in the following that the answer to this question is negative in general. The author is indebted to C. A. Akemann for pointing out the following example of Dixmier-Behncke-Krauss-Leptin [3].

Let H be the incomplete infinite tensor product of H_i, where every H_i

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is a copy of a separable infinite-dimensional Hilbert space H_0. For each integer i we consider the factorization of H,

$$H = H_1 \otimes H_2 \otimes \cdots \otimes H_{i-1} \otimes \left(\bigotimes_{n=i}^{\infty} H_n \right),$$

and write K_x as the C^*-algebra $1 \otimes C(\bigotimes_{n=i}^{\infty} H_n)$ in this factorization, where $C(\bigotimes_{n=i}^{\infty} H_n)$ means the algebra of all compact operators in $\bigotimes_{x=i}^{\infty} H_n$. The C^*-algebra $A = C^*(K_i)$ on H generated by these K_i's is the example mentioned above. This algebra is postliminal. Furthermore, for each ideal I of A there is an integer i_0 such that $I = C^*(K_n;\ n \le i_0)$ and the quotient algebra A/I is isomorphic to the C^*-algebra $A_{i_0+1} = C^*(K_n;\ n \ge i_0 + 1)$.

Let p be an infinite-dimensional projection in H_0 with infinite-dimensional complement $1 - p$. Put

$$p_n = 1 \otimes 1 \otimes \cdots \otimes \overset{n\text{th component}}{p} \otimes 1 \otimes \cdots.$$

Then every p_n induces a derivation δ_n in A. We shall show that the derivation $\delta = \sum_{n=1}^{\infty} \delta_n / 2^{n-1}$ given by the element $h = \sum_{n=1}^{\infty} p_n / 2^{n-1}$ has the required properties. Let ϕ be a bounded linear functional on $B(H_0)$, the algebra of all bounded linear operators on H_0, such that $\langle C(H_0), \phi \rangle = 0$, $\langle 1_{H_0}, \phi \rangle = 0$ and $\langle p, \phi \rangle \neq 0$. Take an element $1 \otimes x$ in K_2 with $x \neq 0$, and choose a bounded linear functional ψ on $B(\bigotimes_{n=2}^{\infty} H_n)$ such that $\langle x, \psi \rangle \neq 0$. Then we have

$$\langle C(H), \phi \otimes \psi \rangle = \left\langle C(H_1) \otimes_{\alpha} C\left(\bigotimes_{n=2}^{\infty} H_n \right),\ \phi \otimes \psi \right\rangle = 0$$

and

$$\langle K_i, \phi \otimes \psi \rangle = 0 \quad \text{for every } i \ge 2.$$

Hence,

$$\langle A,\ \phi \otimes \psi \rangle = 0,$$

whereas

$$\langle h(1 \otimes x),\ \phi \otimes \psi \rangle = \left\langle p \otimes x + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} p_n (1 \otimes x),\ \phi \otimes \psi \right\rangle$$

$$= \langle p \otimes x,\ \phi \otimes \psi \rangle = \langle p, \phi \rangle \langle x, \psi \rangle \neq 0.$$

It follows that h is not a multiplier of A. If k is another generator of δ, then $h - k = \lambda 1_H$ for some scalar λ, because A is irreducible on H. Therefore the derivation δ is not determined by a multiplier of A. Next let I be a (nonzero) closed ideal of A. Then there is an integer i such that $I = C^*(K_n; n \leq i)$ and $A/I \cong A_{i+1}$ as we have mentioned above. The derivation induced in A/I from δ corresponds to the derivation in A_{i+1} given by the element $\sum_{n=i+1}^{\infty} p_n/2^{n-1}$, and a similar argument as above may be applied to show that this derivation is not determined by a multiplier of the quotient algebra A/I.

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