

STRUCTURE DIAGRAMS FOR PRIMITIVE BOOLEAN ALGEBRAS

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ABSTRACT. If S and T are structure diagrams for primitive Boolean algebras, call a homomorphism f from S onto T *right-strong* iff whenever $x T f(t)$, there is an s such that $f(s) = x$ and $s S t$; let RSE denote the category of diagrams and onto right-strong homomorphisms. The relation " S structures \mathfrak{B} " between diagrams and Boolean algebras induces a 1-1 correspondence between the components of RSE and the isomorphism types of primitive Boolean algebras. Up to isomorphism, each component of RSE contains a unique minimal diagram and a unique maximal tree diagram. The minimal diagrams are like those given in a construction by William Hanf. The construction which is given for producing maximal tree diagrams is recursive; as a result, every diagram S structures a Boolean algebra recursive in S .

1. **Right-strong epimorphisms.** We shall use the following notation for binary relations: If P, Q are relations, then $PQ = \{\langle x, z \rangle \mid \exists y: x P y \text{ and } y Q z\}$; $\text{dom } P = \{x \mid \exists y: x P y\}$; $|P| = \{x \mid \exists y: x P y \text{ or } y P x\}$; $P^\sim = \{\langle y, x \rangle \mid x P y\}$; if P is a function, then $P(x) = y$ iff $x P y$; in any case $P[y] = \{x \mid x P y\}$. While this last condition is perhaps unconventional, it is at least compatible with the notation for Boolean algebras: $\mathfrak{U}[a] = \{b \in \mathfrak{U} \mid b \leq a\}$.

A Boolean algebra \mathfrak{B} is *pseudo-indecomposable* iff whenever $\mathfrak{B} = \mathfrak{U} \times \mathfrak{C}$, either $\mathfrak{B} \cong \mathfrak{U}$ or $\mathfrak{B} \cong \mathfrak{C}$. A set $A \subseteq |\mathfrak{B}|$ *disjointly generates* \mathfrak{B} iff each $b \in |\mathfrak{B}|$ is the sum of a pairwise disjoint finite subset of A . \mathfrak{B} is *primitive* iff \mathfrak{B} is pseudo-indecomposable and is disjointly generated by the set of all $b \in \mathfrak{B}$ such that $\mathfrak{B}[b]$ is pseudo-indecomposable.

William Hanf has introduced a notion of "structuring" for primitive Boolean algebras in Definition 4.1 and Lemma 4.3 of [1] which we include here with minor changes for the convenience of the present discussion:

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Definition. Suppose S is a countable binary relation, \mathfrak{B} is a Boolean algebra, and F is a relation whose domain is $|S|$, then S structures \mathfrak{B} with F iff

- (1) The range of F disjointly generates \mathfrak{B} and contains 1 but not 0.
- (2) If $a \leq b$, $s F a$, and $t F b$, then $s S t$ or $s = t$.
- (3) If $a \dot{+} b \leq c$ and $a, b, c \in F\check{[s]}$, then $s S s$.
- (4') If $a \dot{+} b = c$, then $F[a] \cup F[b] \supseteq F[c]$.
- (5') If $s S t$ and $t F c$, then for some $a, b \in |\mathfrak{B}|$, $s F a$, $t F b$, and $a \dot{+} b = c$.

Definition 1. Like Hanf, we will call a countable transitive relation with a largest element a *diagram*. A *diagram homomorphism* $f: S \rightarrow T$ is a function such that $x S y$ implies $f(x) T f(y)$; f is a *right-strong* homomorphism iff whenever $x T f(t)$, there is an s such that $s S t$ and $f(s) = x$. The class of onto right-strong homomorphisms is closed under composition, so we may let RSE be the category of diagrams and onto right-strong homomorphisms.

I am indebted to the referee for pointing out the interesting work of R. S. Pierce [2]. His category \mathcal{Q} , as given in Definitions 8.3 and 8.7, is similar to RSE . In particular, the category of finite diagrams and right-strong homomorphisms is essentially the category of all finite Q.O. systems which have smallest elements. For each finite diagram S , let $<_S = \{ \langle y, x \rangle \mid x S y \text{ or } x = y \}$, and let $P_S = \{ x \in |S| \mid x S x \}$. Then the map $S \mapsto \langle |S|, <_S, P_S \rangle$ induces the indicated isomorphism, as is easily checked.

Lemma 2. Suppose $g: S \rightarrow T$ is an RSE map and S structures \mathfrak{B} with F , then T structures \mathfrak{B} with $g\check{F}$.

Proof. We need to show $g\check{F}$ satisfies conditions (1)–(5'). Conditions (1), (2), and (4') are straightforward. For (3), suppose that $a, b, c \in F\check{[s]}$ and $a \dot{+} b \leq c$. Pick $u, v, w \in |S|$ so that $s g\check{u} F a$, $s g\check{v} F b$, and $s g\check{w} F c$. By (2) and the fact that $a, b \leq c$, we have $u S w$ or $u = w$, and $v S w$ or $v = w$. If $u S w$ or $v S w$, then $s T s$ since $s = g(u) = g(v) = g(w)$. On the other hand, if $u = v = w$, then $a, b, c \in F\check{[u]}$ and by (3), $u S u$, and thus $s T s$. For (5'), suppose that $s T t$ and $t g\check{F} c$; choose $v \in |S|$ so that $t g\check{v} F c$. Since g is right-strong, we can also choose $u \in |S|$ so that $g(u) = s$ and $u S v$. Then for some $a, b \in |\mathfrak{B}|$, $u F a$, $v F b$, and $a \dot{+} b = c$, by (5') applied to S ; and, of course, $s g\check{F} a$ and $t g\check{F} b$. \square

Although the converse of the above lemma is false, the following lemma is in the same spirit, and leads to an alternate proof of the fact that every diagram structures a Boolean algebra (see Hanf's Theorem 7.2).

Definition 3. If S is a diagram, let $S^+ = S \cup \{ \langle x, x \rangle \mid x \in |S| \}$, and let $\mathfrak{U}(S)$, the ideal algebra on S , be the Boolean algebra of sets generated by $\{S^+[x] \mid x \in |S|\}$. S is an *irreflexive tree diagram* iff $\forall x \in |S|$, not $x S x$, and $\forall x, y, z \in |S|$, $(x S y \text{ and } x S z) \text{ imply } (y S^+ z \text{ or } z S^+ y)$.

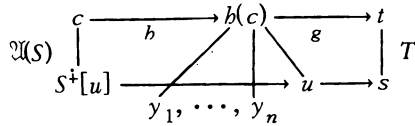
Definition 4. An RSE map $g: S \rightarrow T$ has *large collapse* iff $\forall x \in |S|$, no finite nonempty antichain A in $S[x]$ is maximal in $S[x] \cap g^{-1}[A]$.

Lemma 5. Suppose $g: S \rightarrow T$ has large collapse and S is an irreflexive tree diagram, then T with $g \sim h \sim$ structures $\mathfrak{U}(S)$, where $\forall x \in |S|$, $\forall y_1, \dots, y_n \in S[x]$, $h(S^+[x] - (S^+[y_1] \cup \dots \cup S^+[y_n])) = x$.

Proof. It is easy to verify that S with $h \sim$ satisfies conditions (1)–(4’); thus so does T with $g \sim h \sim$, as in the previous lemma. For (5’), suppose $s T t$ and $t g \sim h \sim c$; then for some $y_1, \dots, y_n \in S[h(c)]$,

$$c = S^+[h(c)] - (S^+[y_1] \cup \dots \cup S^+[y_n]).$$

We may assume further that $\{y_1, \dots, y_n\}$ is an antichain since S is tree ordered. We wish to choose u so that $u S h(c)$, $g(u) = s$, and u is not S -related to any y_i :



Let A be the set of y_i 's for which $s T^+ g(y_i)$. If A is empty, we can choose u so that $u S h(c)$, $g(u) = s$, and no y_i is below u , since S is a tree diagram and $g[S] \cap S[h(c)]$ is infinite by the strong collapse property. Such a point u will not lie below any y_i either, since otherwise $u S y_i$ implies $s T^+ g(y_i)$. If A is not empty, it can be extended to an infinite antichain $A' \subseteq S[h(c)] \cap g^{-1}[A]$, by strong collapse. Since S is a tree we can choose $v \in A' - A$ so that no y_i belongs to $S[v]$. v is not below any y_i either, for suppose $v S^+ y_i$: because $v \in A'$, there is some $y_j \in A$ such that $g(v) = g(y_j)$, so that $g(y_j) T^+ g(y_i)$. Since $y_j \in A$, $s T^+ g(y_j)$, and thus $s T^+ g(y_i)$, so that $y_i \in A$; but v is not S -related to any element of A , a contradiction. Either $s = g(v)$ or $s T g(v)$ by choice of v . If $s T g(v)$, pick u so that $g(u) = s$ and $u S v$; if $s = g(v)$, let $u = v$. Then since S is a tree, u is not S -related to any y_i either, and $u S^+ v S h(c)$. From this it is clear that $c = (c - S^+[u]) \dot{+} S^+[u]$, and of course $g(h(S^+[u])) = g(u) = s$, and $g(h(c - S^+[u])) = g(h(c)) = t$. \square

Construction 6. Every diagram S is the range of an RSE map $\mu: MT(S) \rightarrow S$ with large collapse, where $MT(S)$ is an irreflexive tree diagram, and μ and $MT(S)$ are recursive in S .

Proof. Let S be a diagram. Let R be the irreflexive diagram given by $|R| = \{\langle x, j \rangle \mid x S x, j \in \omega\} \cup \{\langle x, 0 \rangle \mid \text{not } x S x\}$, and $\langle x, j \rangle R \langle y, k \rangle$ iff $(x S y \text{ and not } y S x)$ or $(x S y \text{ and } y S x \text{ and } k < j)$. Let $p_1: R \rightarrow S$ be the first coordinate projection. Then R is irreflexive, and p_1 is right-strong. Let $|Q| = \{\langle 1_R, 0 \rangle\} \cup (|R| - \{1_R\}) \times \omega$ and let $\langle x, j \rangle Q \langle y, k \rangle$ iff $x Q y$; let $p_2: \tilde{Q} \rightarrow R$ be the first coordinate projection. Then p_2 is right-strong and has large collapse. Finally, let $|MT(S)|$ be the set of all finite Q -chains with largest element 1; demand of $MT(S)$ only that whenever $\langle x_1, \dots, x_j, \dots, 1 \rangle$ is a Q -chain, we have $\langle x_1, \dots, 1 \rangle MT(S) \langle x_j, \dots, 1 \rangle$. Let $p_3: MT(S) \rightarrow Q$ be given by $p_3(x_1, \dots, 1) = x_1$. Then $MT(S)$ has the prescribed properties, and p_3 is right-strong. Let $\mu = p_1 p_2 p_3$; the large collapse condition carries over from p_2 to μ . Finally, it is clear from the construction that μ and $MT(S)$ are recursive in S . \square

Corollary 7. *If S structures a Boolean algebra \mathfrak{B} , then S structures \mathfrak{B} with $f \sim$ for some function f .*

Proof. S structures $\mathfrak{U}(MT(S))$ with $\mu \sim b \sim$, where b and μ are as above; by Hanf's Theorem 4.5, $\mathfrak{B} \cong \mathfrak{U}(MT(S))$. \square

Corollary 8. *Every diagram S structures a Boolean algebra recursively definable in S .*

Proof. It suffices to construct an algebra isomorphic with $\mathfrak{U}(MT(S))$ that is recursive in $MT(S)$. Let \mathfrak{B} be the set of all pairs $\langle a, A \rangle$, where A is a finite antichain in $MT(S)[a]$. Then $\forall c \in h[MT(S)]$, $\exists! \langle a, A \rangle \in \mathfrak{B}$: $c = MT(S)^+[a] - \bigcup \{MT(S)^+[x] \mid x \in A\}$. Let $|\mathfrak{B}|$ be the set of all finite subsets $\{\langle a_1, A_1 \rangle, \dots, \langle a_n, A_n \rangle\}$ of \mathfrak{B} such that if for some i, j , $a_i MT(S)^+ a_j$; then for some $b \in A_j$, $a_i MT(S) b$. Each element of $\mathfrak{U}(MT(S))$ is uniquely represented as the disjoint sum of elements of $h[MT(S)]$, and thus is uniquely represented by an element of $|\mathfrak{B}|$. The recursiveness in $MT(S)$ of the Boolean operations on \mathfrak{B} is now straightforward. To compute the join operation, for example, first define the meet of two finite antichains A, B , by $A \wedge B = \{x \in A \cup B \mid x \in A \cap B \text{ or } \exists y \in A \cup B, x MT(S) y\}$. Then to find the join of a finite subset $\{\langle a_1, A_1 \rangle, \dots, \langle a_n, A_n \rangle\}$ of \mathfrak{B} , just replace each pair $\langle a_i, A_i \rangle$ and $\langle a_j, A_j \rangle$ with $\langle a_j, A_i \wedge A_j \rangle$ whenever $a_i MT(S)^+ a_j$ and $\forall b \in A_j$, not $a_i MT(S) b$. Finally, the join of two elements from \mathfrak{B} is the join of their union. \square

2. Minimal and maximal diagrams. Consider RSE to be preordered by the relation $Q \leq R$ iff there is an RSE map from R to Q . The RSE

minimal diagrams turn out to be just those of the type $S(\mathfrak{B})$ given in Hanf's Construction 5.3, which for convenience, we restate as follows: Suppose \mathfrak{B} is a primitive Boolean algebra. Let $PI(\mathfrak{B}) = \{b \in |\mathfrak{B}| \mid \mathfrak{B}[b] \text{ is pseudo-indecomposable}\}$. $\forall a \in PI(\mathfrak{B})$, let $\tau(a) = \{b \mid \mathfrak{B}[b] \cong \mathfrak{B}[a]\}$. Let $|S(\mathfrak{B})| = \{\tau(a) \mid a \in PI(\mathfrak{B})\}$, and $\forall a, b \in PI(\mathfrak{B})$, let $\tau(a) S(\mathfrak{B}) \tau(b)$ iff $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. The following result is analogous to Pierce's Lemma 9.1.

Theorem 9. *If a diagram R structures a Boolean algebra \mathfrak{B} , then there is a unique RSE map π from R to $S(\mathfrak{B})$.*

Proof. Assume R structures \mathfrak{B} with F . We shall show that $\pi = F\tau$ is the required RSE map from R to $S(\mathfrak{B})$.

Step I. $\pi = F\tau$ is well defined: Suppose $s F a$ and $s F b$; then by Hanf's Lemma 4.4, $\mathfrak{B}[a] \cong \mathfrak{B}[b]$. By the proof of Hanf's Theorem 5.2, $a, b \in PI(\mathfrak{B})$, and thus $\tau(a) = \tau(b)$.

Step II. π is onto; that is, if $b \in PI(\mathfrak{B})$, then $\exists b' \in PI(\mathfrak{B})$, $\exists s \in |R|$, $s F b'$ and $\tau(b) = \tau(b')$: By condition (1) of the structure definition, $b = b_1 \dot{+} \dots \dot{+} b_n$ with b_1, \dots, b_n in the range of F , and for some b_i , $\mathfrak{B}[b] \cong \mathfrak{B}[b_i]$ since $\mathfrak{B}[b]$ is pseudo-indecomposable.

Step III. π is a homomorphism; that is, if $s R t$, then for some $a, b \in |\mathfrak{B}|$, $s F a$, $t F b$, and $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. Pick c so that $t F c$. Then by condition (5') of the structure definition, we can choose a, b so that $s F a$, $t F b$, and $a \dot{+} b = c$; then by Hanf's Lemma 4.4,

$$\mathfrak{B}[b] \cong \mathfrak{B}[c] = \mathfrak{B}[a \dot{+} b] \cong \mathfrak{B}[a] \times \mathfrak{B}[b].$$

Step IV. π is right-strong: Suppose $\pi(s) S(\mathfrak{B}) \pi(t)$. Then we may choose a, b so that $s F a$ and $t F b$; in which case $\tau(a) = \pi(s) S(\mathfrak{B}) \pi(t) = \tau(b)$, so that $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. Hence we may write $b = a_1 \dot{+} b_1$, where $\tau(a_1) = \tau(a)$ and $\tau(b_1) = \tau(b)$. By condition (1) of the structure definition we may write $a_1 = a_2 \dot{+} c$ and $b_1 = b_2 \dot{+} d$, where $\tau(a_1) = \tau(a_2)$, $\tau(b_1) = \tau(b_2)$, and a_2, b_2 are in the range of F . Choose s', t' so that $s' F a_2$ and $t' F b_2$; then $\pi(s') = \tau(a) = \pi(s)$ and $\pi(t') = \tau(b) = \pi(t)$. Since $b \geq a_2 \dot{+} b_2$, we have $(s' R t$ or $s' = t)$ and $(t' R t$ or $t' = t)$ by condition (2) of the structure definition. If $s' R t$, we are done; suppose $s' = t$. If $t' = t$, then since $b \geq a_2 \dot{+} b_2$, we have $s' = t R t$ by condition (3). If $t' R t$, however, set $b_2 = b_3 \dot{+} a_3$, with $\tau(b_2) = \tau(b_3) = \tau(b)$ and $\tau(a_3) = \tau(a)$; as before, set $a_3 = a_4 \dot{+} a_5$, with $\tau(a_3) = \tau(a_4)$, and a_4 in the range of F . Pick s'' so that $s'' F a_4$. Then by condition (2), $s'' R t'$ or $s'' = t'$, since $a_4 \leq b_2$. In either case $s'' R t$, and $\pi(s'') = \tau(a_4) = \tau(a) = \pi(s)$.

Step V. π is unique: First, $\pi = F\tau$ is independent of F ; that is, if R structures \mathfrak{B} with F and with G , then $F\tau = G\tau$: Suppose $s F a$ and $s G b$; then by Hanf's Lemma 4.4, $\mathfrak{B}[a] \cong \mathfrak{B}[b]$, and thus $\tau(a) = \tau(b)$. Next, if $S(\mathfrak{B})$ structures \mathfrak{B} with H , then $H \subseteq \tau^\sim: S(\mathfrak{B})$ structures \mathfrak{B} with τ^\sim , as is shown in the proof of Hanf's Theorem 5.4. Also, $H\tau$ is independent of H , by what we have just seen (in the special case where $R = S(\mathfrak{B})$). Hence $H\tau = \tau^\sim\tau$. But $\tau^\sim\tau$ is the identity map on $S(\mathfrak{B})$, and thus $H \subseteq \tau^\sim$. Finally, if g is an RSE map from R to $S(\mathfrak{B})$, then $g = \pi$: By Lemma 2, $S(\mathfrak{B})$ with $g^\sim F$ structures \mathfrak{B} . Hence $g^\sim F \subseteq \tau^\sim$, in which case $F^\sim g \subseteq \tau$, and thus $g \subseteq F F^\sim g \subseteq F\tau = \pi$. But g and π are functions with the same domain; so $g = \pi$. \square

Corollary 10. *The RSE minimal diagrams are just those of the type given in Hanf's Construction 5.3. They are essentially the irreducible simple P.O. systems of Pierce.*

Proof. For the first statement, suppose that $f: S(\mathfrak{B}) \rightarrow R$ is an RSE map. Then R structures \mathfrak{B} by Lemma 2 and Hanf's Theorem 5.4. Hence, we need only let $\pi: R \rightarrow S(\mathfrak{B})$ be the unique map given above. Notice that $f\pi$, being the unique RSE map from $S(\mathfrak{B})$ to $S(\mathfrak{B})$, is the identity; consequently, since f is onto, it is an isomorphism. The second statement now follows from the definitions supplied with Pierce's Propositions 8.13 and 8.14, and the fact that every right-strong homomorphism factors through an RSE map. \square

Corollary 11. *The relation "S structures \mathfrak{B} " induces a 1-1 correspondence between the components of RSE and the isomorphism types of primitive Boolean algebras.*

Proof. If S and T structure the Boolean algebra \mathfrak{B} , then by the above theorem, both precede $S(\mathfrak{B})$ in RSE. Conversely, suppose $f: Q \rightarrow R$ is an RSE map. If Q structures \mathfrak{B} , then so does R by Lemma 2. On the other hand, if R structures \mathfrak{B} , then Q structures $\mathfrak{U}(MT(Q))$ by Construction 6, R structures $\mathfrak{U}(MT(Q))$ by Lemma 2, and $\mathfrak{U}(MT(Q)) \cong \mathfrak{B}$ by Hanf's Lemma 4.4; hence Q also structures \mathfrak{B} . A finite repetition of this argument shows that if S and T lie in the same component of RSE, they structure the same Boolean algebras. \square

Lemma 12. *Suppose R structures \mathfrak{B} , $\pi: R \rightarrow S(\mathfrak{B})$ is the unique RSE map, $x, y \in |R|$, and $R[x] \cong R[y]$ (or more accurately, $R|R[x] \cong R|R[y]$). Then $\pi(x) = \pi(y)$.*

Proof. $\forall x, y \in |R|$, set $x \sim y$ iff $R[x] \cong R[y]$. Then the projection map from R to R/\sim is easily seen to be right-strong. Consequently, it must factor through π . \square

Theorem 13. *For any primitive Boolean algebra \mathfrak{B} , there is (up to isomorphism) a unique tree diagram P that structures \mathfrak{B} and satisfies the following equivalent conditions:*

- (i) P is RSE maximal.
- (ii) P is of the type given in Construction 6.
- (iii) P is irreflexive, and $\forall x \in |P|$, $P^\sim[x]$ is finite and $\forall y \in P[x]$, there exist infinitely many $z \in P[x]$ such that $P[z]$ is a maximal ideal in $P[x]$ isomorphic with $P[y]$.

Proof. The diagrams discussed will structure \mathfrak{B} , unless otherwise indicated. The first step is to show that if P is a tree diagram, S satisfies (iii), and $f: P \rightarrow S$ is an RSE map, then $P \cong S$: First, P must be irreflexive since S is. Also, $\forall x$, $f|P^\sim[x]$ must be an embedding, and thus $P^\sim[x]$ is finite. For each $s \in |S|$, let $B(s)$ be the set of all $s' \in |S|$ such that $S[s'] \cong S[s]$, and s and s' have the same S -successor. For each $x \in |P|$, let $A(x)$ be the set of all $x' \in |P|$ such that $S[f(x)] \cong S[f(x')]$ and x and x' have the same P -successor. We can define an isomorphism $g: P \rightarrow S$ inductively as follows: let $g(1) = 1$. Assume that for $n > 0$, g has been defined on $D_n = \{y | (P^\sim[y])^\sim = n\}$, and that for each $y \in D_n$, $S[g(y)] \cong S[f(y)]$. Extend g to each x such that $(P^\sim[x])^\sim = n$ as follows: Suppose y is the P -successor of x ; then $f(x) S f(y)$. By (iii) we may choose s_0 so that $S[s_0] \cong S[f(x)]$ and $f(y)$ is the S -successor of s_0 . By assumption, $S[f(y)] = S[g(y)]$, and thus we may choose s so that $S[s] \cong S[s_0]$ and $g(y)$ is the S -successor of s . $B(s)$ is infinite by (iii). So is $B(s_0)$; for each $s' \in B(s_0)$, there is an $x' \in P$ such that $f(x') = s'$ and $x' P y$, since $s' S f(y)$ and f is right-strong; furthermore, y is the P -successor of x' since $f(y)$ is the S -successor of s' . Each such x' belongs to $A(x)$; hence $A(x)$ is infinite. Extend g to $A(x)$ in any way such that $g|A(x)$ is a 1-1 correspondence onto $B(s)$. In this way g may be extended to the set of all x such that $(S^\sim[x])^\sim = n$ since both of the families $\{A(x) | x \in |P|\}$ and $\{B(s) | s \in |S|\}$ are pairwise disjoint. The completed map $q: P \rightarrow S$ so obtained will be an isomorphism by construction.

A moment's thought shows that the diagrams given in Construction 6 satisfy (iii); thus (ii) \rightarrow (iii). The above paragraph shows directly that (iii) \rightarrow (i). Suppose P is maximal; then in RSE, P precedes $MT(P)$ since $MT(P)$ precedes P , and thus $P \cong MT(P)$ since $MT(P)$ satisfies (iii);

hence (i)→(ii). Finally, uniqueness may be shown as follows: Suppose Q, R satisfy (iii), let $S = S(\mathcal{B})$, and let $\pi: Q \rightarrow S$ and $\rho: R \rightarrow S$ be the unique RSE maps. We may define an isomorphism $\phi: Q \rightarrow R$ by a back and forth argument as follows: Since $\forall x \in |Q|, Q^\vee[x]$ is finite, we may let $\{x_i | i \in \omega\}$ be an enumeration of $|Q|$ such that $\forall x_i, Q^\vee[x_i] \subseteq \{x_j | j < i\}$. Let $\{y_i | i \in \omega\}$ be a similar enumeration of $|R|$. Let $\phi_0(1_Q) = 1_R$. Assume a partial isomorphism ϕ_j from a finite subset of $|Q|$ to $|R|$ has been defined, that for each $x \in \text{dom } \phi_j, \phi_j$ maps $Q^\vee[x]$ onto $R^\vee[\phi_j(x)]$, and that $\pi(x) = \rho(\phi_j(x))$. To extend ϕ_j to the next element, say $x_i \in |Q|$ for example, let x be the Q -successor of x_i . Then $\pi(x_i) S \pi(x) = \rho(\phi_j(x))$, and we may choose $y \in |R|$ so that $y R \phi_j(x)$ and $\rho(y) = \pi(x_i)$. Then there are infinitely many points $z \in R[\phi_j(x)]$ such that $\phi_j(x)$ is the R -successor of z and $R[y] \cong R[z]$; let $\phi_{j+1}(x_i)$ be one such z that does not belong to the range of ϕ_j . Since $R[\phi_{j+1}(x_i)] \cong R[y], \rho(\phi_{j+1}(x_i)) = \rho(y) = \pi(x_i)$, by the above lemma; ϕ_{j+1} maps $Q^\vee[x_i]$ onto $R^\vee[\phi_{j+1}(x_i)]$, and it is clearly a partial isomorphism. ϕ , the union of the ϕ_j 's is the desired isomorphism. \square

Pierce's Proposition 10.2 suggests the following analogue for diagrams and primitive Boolean algebras: Define the *product* of two diagrams S, T by $(s, t) S \times T (s', t')$ iff $(s S^+ s'$ and $t T^+ t')$ and $(s S s'$ or $t T t')$. For Boolean algebras \mathcal{U}, \mathcal{B} , let $\mathcal{U} \oplus \mathcal{B}$ denote the coproduct of \mathcal{U} and \mathcal{B} in the category of Boolean algebras and homomorphisms that preserve 1.

Proposition 14. *If S structures \mathcal{U} and T structures \mathcal{B} , then $S \times T$ structures $\mathcal{U} \oplus \mathcal{B}$.*

Proof. Suppose S structures \mathcal{U} with F and T structures \mathcal{B} with G . We may consider $\mathcal{U} \oplus \mathcal{B}$ to consist of all formal sums $a_1 b_1 \dot{+} \dots \dot{+} a_n b_n$ where $\forall i < j$, either $a_i a_j = 0$ or $b_i b_j = 0$, modulo the usual identities for Boolean algebras. It is now a straightforward task to show that $S \times T, F \times G$, and $\mathcal{U} \oplus \mathcal{B}$ satisfy the structure definition, except perhaps for condition (5'): Assume that $(s, t) S \times T (s', t')$ and $(s', t') F \times G a'' b''$. Then $s' S a'', t' G b''$, and $(s S s'$ and $t T^+ t')$ or vice versa. If $s S s'$ and $t = t'$, then there are $a, a' \in |\mathcal{U}|$ such that $s F a, s' F a'$, and $a \dot{+} a' = a''$. But then $(s, t) F \times G (a, b''), (s', t') F \times G (a', b'')$, and $ab'' \dot{+} a'b'' = a''b''$. In the other interesting case, if $s S s'$ and $t T t'$, then there are $a, a' \in |\mathcal{U}|$ and $b, b' \in |\mathcal{B}|$ such that $s F a, s' F a', t G b, t' G b', a \dot{+} a' = a'',$ and $b \dot{+} b' = b''$, so that

$$ab \dot{+} a'b' \leq ab \dot{+} a'b \dot{+} ab' + a'b' = a''b'',$$

and $(s, t) F \times G ab$ and $(s', t') F \times G a'b'$; by Hanf's Lemma 4.3, this is good enough. \square

The above result is interesting because it suggests that diagrams structure Boolean algebras contravariantly (as does the pairing of the embedding $S[s] \subseteq S[1]$ with the projection $\mathfrak{A} \rightarrow \mathfrak{A}[a]$, where S structures \mathfrak{A} and $S[S[s]]$ structures $\mathfrak{A}[a]$). As with Pierce's products of P.O. systems, the above diagram product has not yet been placed in a correct categorical framework. Perhaps there is a product-preserving contravariant functor from a category containing RSE to a suitable category of isomorphism types of Boolean algebras.

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