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ON SATURATED FORMATIONS WHICH ARE SPECIAL RELATIVE TO THE STRONG COVERING-AVOIDANCE PROPERTY

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ABSTRACT. Let $\mathfrak{F}$ be a saturated formation of finite soluble groups. Let G be a finite soluble group and F an $\mathfrak{F}$-projector of G. Then F is said to satisfy the strong covering-avoidance property if (i) F either covers or avoids each chief factor of G, and (ii) $F \cap L/F \cap K$ is a chief factor of F whenever L/K is a chief factor of G covered by F. Let $\mathcal{C}_{\mathfrak{F}}$ denote the class of all finite soluble G in which the $\mathfrak{F}$-projectors satisfy the strong covering-avoidance property. $\mathcal{C}_{\mathfrak{F}}$ is a formation. Let $\mathcal{Y}_{\mathfrak{F}}$ be the class of groups G in which an $\mathfrak{F}$-normalizer is also an $\mathfrak{F}$-projector. $\mathcal{Y}_{\mathfrak{F}}$ is a formation studied by Klaus Doerk. Note that $\mathcal{Y}_{\mathfrak{F}} \subseteq \mathcal{C}_{\mathfrak{F}}$. $\mathfrak{F}$ is said to be $\mathcal{C}$-special if $\mathcal{C}_{\mathfrak{F}} = \mathcal{Y}_{\mathfrak{F}}$. The purpose of this note is to study $\mathcal{C}$-special formations. Two characterizations of $\mathcal{C}$-special formations are given. Let i be a positive integer and let $\mathfrak{N}^{(i)}$ denote the class of finite soluble groups G whose Fitting length is at most i. Then $\mathfrak{N}^{(i)}$ is $\mathcal{C}$-special. Finally, the formation $\mathcal{C}_{\mathfrak{F}}$ is saturated if and only if $\mathfrak{F}$ is the class of all finite soluble groups.

Let $\mathfrak{F}$ be a saturated formation which contains the class $\mathfrak{N}$ of finite nilpotent groups as a subformation and let $\mathcal{P}$ be a property which a subgroup of a finite group may possess. The class of all finite soluble groups in which the $\mathfrak{F}$-projectors have property $\mathcal{P}$ is denoted by $\mathcal{P}_{\mathfrak{F}}$. $\mathfrak{F}$ is said to be special relative to $\mathcal{P}$ or simply $\mathcal{P}$-special if $\mathcal{P}_{\mathfrak{F}}$ coincides with the formation $\mathcal{Y}_{\mathfrak{F}}$ of all finite soluble groups in which the $\mathfrak{F}$-normalizers are also the $\mathfrak{F}$-projectors.

Let $\mathcal{P}$ be defined as follows: A subgroup H of G is said to have property $\mathcal{P}$ if $G = C_G(R/S)H$ whenever R/S is a chief factor of G covered by H. One can easily check that in this case $\mathcal{P}_{\mathfrak{F}} = \mathcal{B}_{\mathfrak{F}}$, the class of all finite soluble groups G in which the $\mathfrak{F}$-projectors cover only the $\mathfrak{F}$-central chief factors of G (see Doerk [2, Definition 4.1 (a)]). In [2], Doerk characterizes

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those saturated formations which are special relative to this property $\mathcal{P}$. In particular, he shows (cf. [2, Satz 4.11]) that $\mathfrak{F}$ is $\mathcal{P}$-special if and only if $\mathfrak{F} = S_{p'}\mathfrak{F}(p)$ for some prime p, where $\{\mathfrak{F}(q)\}$ is the canonical definition of $\mathfrak{F}$ and $S_{p'}$ is the class of all finite soluble p'-groups. He also characterizes (cf. [2, Satz 4.9]) those saturated formations $\mathfrak{F}$ for which $\mathcal{P}_{\mathfrak{F}} = \mathcal{B}_{\mathfrak{F}}$.

The aim of this paper is to investigate the $\mathcal{C}$-special saturated formations, where $\mathcal{C}$ denotes the strong covering-avoidance property. A subgroup H of a finite group G is said to have the *strong covering-avoidance property* if (i) H either covers or avoids each chief factor of G, and (ii) $H \cap L/K \cap L$ is a chief factor of H whenever L/K is a chief factor of G covered by H. In view of Theorem 4.1 of Carter and Hawkes [1], an $\mathfrak{F}$-normalizer of any finite soluble group has the strong covering-avoidance property. Hence, $\mathcal{Y}_{\mathfrak{F}}$ is always contained in $\mathcal{C}_{\mathfrak{F}}$.

In §2, we discuss a simple characterization and give some examples of saturated formations which are $\mathcal{C}$-special.

In §3, we show that, unlike the case of the property $\mathcal{P}$ defined above, the only $\mathcal{C}$-special saturated formations $\mathfrak{F} \neq \{1\}$ for which $\mathcal{C}_{\mathfrak{F}}$ is saturated is the class of all finite soluble groups.

Throughout this paper only finite soluble groups are considered. $\mathfrak{F}$ will always denote a saturated formation containing $\mathfrak{N}$. Further, $\{\mathfrak{F}(p)\}$ will always denote the canonical definition of $\mathfrak{F}$; that is for each prime p, $S_p\mathfrak{F}(p) = \mathfrak{F}(p) \subseteq \mathfrak{F}$ and $\{\mathfrak{F}(p)\}$ locally define $\mathfrak{F}$. Also, for each prime p, $\mathfrak{F}^*(p)$ denotes, as in Doerk [2], the class of all groups whose $\mathfrak{F}$-projectors belong to $\mathfrak{F}(p)$. $\mathfrak{F}^*(p)$ is a formation. A p-chief factor H/K of the group G is called $\mathfrak{F}^*$-*central* if $G/C_G(H/K) \in \mathfrak{F}^*(p)$. If H/K is not $\mathfrak{F}^*$-*central*, then it is termed $\mathfrak{F}^*$-*eccentric*. The notation used is standard. For various basic definitions and terminology unexplained here we refer the readers to Carter and Hawkes [1], Gaschütz [4] and Huppert [6].

2. A characterization and some examples of $\mathcal{C}$-special formations.

We begin this section by showing

Proposition 1. *$\mathcal{C}_{\mathfrak{F}}$ is a formation.*

For the proof of Proposition 1 we need the following.

Lemma 2. *Let G be a group and V an $\mathfrak{F}$-projector of G. If V covers or avoids each chief factor in a given chief series of G and intersects the latter into a chief series of V, then V has the strong covering-avoidance property.*

Proof. By induction on $|G|$. We first show that V has the covering-avoidance property in G. Let A be the minimal normal subgroup of G in the given chief series of G, and let H/K be an arbitrary chief factor of G. If H/K is $\mathfrak{F}^*$-central, then, by a result of Dade (cf. [2, Satz 2.9]), V covers H/K. Hence, we can assume that H/K is $\mathfrak{F}^*$-eccentric. First, assume that A avoids H/K. Then AH/AK is an $\mathfrak{F}^*$-eccentric chief factor of G, hence avoiding V. By induction VA/A either covers or avoids a chief factor of G/A. By Huppert [7, Satz 2.1],

$$V \cap H \subseteq V \cap AH \subseteq V \cap AK = (V \cap A)(V \cap K),$$

so that,

$$V \cap H = (V \cap K)(V \cap H \cap A) \subseteq K$$

since A avoids H/K. Thus, V avoids H/K in this case.

Suppose next that A covers H/K. Then A is G-isomorphic to H/K, and so, A is $\mathfrak{F}^*$-eccentric. Hence, $V \cap A = 1$. Because of Huppert [7, Satz 2.1], we have

$$V \cap H = V \cap KA = (V \cap K)(V \cap A) = V \cap K$$

and V avoids H/K.

It remains to show that $V \cap H/V \cap K$ is a chief factor of V whenever H/K is a chief factor of G covered by V. If A avoids H/K, then HA/KA is a chief factor of G/A which is covered by VA/A, and so, by induction, $HA \cap VA/KA \cap VA$ is a chief factor of VA. But

$$\begin{aligned} HA \cap VA/KA \cap VA &= A(V \cap H)/A(V \cap K) \\ &\cong (H \cap V/A \cap V)/(K \cap V/A \cap V) \end{aligned}$$

whence $H \cap V/K \cap V$ is a chief factor of V. Hence, assume that A covers H/K. Then A is G-isomorphic to H/K, hence by [2, Satz 2.9] V covers A. By assumption, A is a chief factor of V, hence $H \cap V/K \cap V$ is a chief factor of V. This completes the proof.

Proof of Proposition 1. We show first that $\mathcal{C}_{\mathfrak{F}}$ is a homomorph. Let $G \in \mathcal{C}_{\mathfrak{F}}$, $N \trianglelefteq G$ and let F be an $\mathfrak{F}$-projector of G. Let $(H/N)/(K/N)$ be a chief factor of G/N, then H/K is a chief factor of G, and so, FN/N either covers or avoids $(H/N)/(K/N)$. Assume that FN/N covers $(H/N)/(K/N)$. Then F covers H/K, hence $H \cap F/K \cap F$ is a chief factor of F. Thus,

$$(H \cap F/N \cap F)/(K \cap F/N \cap F)$$

is a chief factor of $F/N \cap F$ and so $(H \cap FN/N)/(K \cap FN/N)$ is a chief factor of FN/N. Hence, $G/N \in \mathcal{C}_{\mathfrak{F}}$.

Next, we show: $G \in \mathcal{C}_{\mathfrak{F}}$ whenever G has two distinct minimal normal subgroups N and M such that both G/N and G/M belong to $\mathcal{C}_{\mathfrak{F}}$. Let F be an $\mathfrak{F}$-projector of G and let H/K be a chief factor of G above N. Since $G/N \in \mathcal{C}_{\mathfrak{F}}$, FN/N either covers or avoids $(H/N)/(K/N)$, and if it covers it then $(H \cap FN/N)/(K \cap FN/N)$ is a chief factor of FN/N. Clearly, F either covers or avoids H/K, and if it covers H/K, then since

$$(H \cap FN/N)/(K \cap FN/N) \cong H \cap F/K \cap F,$$

$H \cap F/K \cap F$ is a chief factor of F. Similarly, since $G/M \in \mathcal{C}_{\mathfrak{F}}$, FM/M either covers or avoids NM/M, and if it covers it then NM/M is a chief factor of FM/M. If FM/M avoids NM/M, then

$$F \cap N \subseteq (F \cap NM) \cap N \subseteq M \cap N = 1,$$

and so F avoids N. On the other hand, if FM/M covers NM/M, then $NM \subseteq FM$, and so, $NM = M(F \cap NM) = M(F \cap N)$ by Huppert [7, Satz 2.1]. An order argument now shows that $N = N \cap F$ and so F covers N. Moreover, N is a chief factor of F since NM/M is a chief factor of FM/M. The proposition now follows from Lemma 2.

We now give a simple characterization of $\mathcal{C}$-special formations. For this purpose we introduce the following definition. The group G is said to be $\mathfrak{F}^*(p)$-*irreducible* if either $G = \{1\}$ or (i) $G \in \mathfrak{F}^*(p) \cap \mathcal{C}_{\mathfrak{F}}$ and (ii) G has a faithful, irreducible $Z_p[G]$-module V such that V_F, the restriction of V to F, an $\mathfrak{F}$-projector of G, is an irreducible $Z_p[F]$-module.

For each prime p, let $\alpha(p)$ denote the class of $\mathfrak{F}^*(p)$-irreducible groups and $A(p)$ the formation generated by $\alpha(p)$.

Remark. Let p be a prime. Then $\alpha(p) \subseteq \mathfrak{F}(p)$ if and only if $\mathfrak{F}(p) = S_pA(p)$.

Proof. Assume that $\alpha(p) \subseteq \mathfrak{F}(p)$ and suppose that $\mathfrak{F}(p) \neq S_pA(p)$. Let $G \in \mathfrak{F}(p) \backslash S_pA(p)$ be of minimal order. Then G has a unique minimal normal subgroup N such that $G/N \in S_pA(p)$. Hence, N is an elementary abelian q-group, where q is a prime distinct from p. By Doerk [2, Hilfssatz 1.3], G has a faithful, irreducible $Z_p[G]$-module V, whence $G \in \alpha(p) \subseteq S_pA(p)$, which is a contradiction. Hence, $\mathfrak{F}(p) = S_pA(p)$.

The converse is immediate.

Theorem 3. *$\mathfrak{F}$ is $\mathcal{C}$-special if and only if $\alpha(p) \subseteq \mathfrak{F}(p)$ for each prime p.*

Proof. Assume that $\mathfrak{F}$ is $\mathcal{C}$-special and let $\{1\} \neq G \in \alpha(p)$. Let V be a faithful, irreducible $Z_p[G]$-module such that V_F, where F is an $\mathfrak{F}$-projector of G, is irreducible. Let $H = [V]G$ be the semidirect product of V by G. Since $G \in \mathfrak{F}^*(p)$, it follows by Doerk [2, Hilfssatz 2.6] that VF is an $\mathfrak{F}$-projector of H. Further, since $G \in \alpha(p)$, $H \in \mathcal{C}_{\mathfrak{F}}$. By our assumption, $\mathcal{C}_{\mathfrak{F}} = \mathcal{Y}_{\mathfrak{F}}$, so that V is $\mathfrak{F}$-central in H. Hence, $G \cong H/V \in \mathfrak{F}(p)$ and so $\alpha(p) \subseteq \mathfrak{F}(p)$.

Conversely, assume that $\alpha(p) \subseteq \mathfrak{F}(p)$ for each prime p. Further, assume that $\mathcal{C}_{\mathfrak{F}} \neq \mathcal{Y}_{\mathfrak{F}}$ and let $G \in \mathcal{C}_{\mathfrak{F}} \backslash \mathcal{Y}_{\mathfrak{F}}$ be of minimal order. Then G has a unique minimal normal subgroup N and $G/N \in \mathcal{Y}_{\mathfrak{F}}$. Since $G \notin \mathcal{Y}_{\mathfrak{F}}$, an $\mathfrak{F}$-projector F of G covers N. Let $|N| = p^n$, $n > 0$, and assume that $C_G(N) \neq N$. Let $L = G/C_G(N)$ and let $H = [N]L$, the semidirect product of N by L. By Huppert [6, Hilfssatz 7.21, p. 70], $H \in \mathcal{C}_{\mathfrak{F}}$. Since $|H| < |G|$, $H \in \mathcal{Y}_{\mathfrak{F}}$. Hence $L \cong H/N \in \mathfrak{F}(p)$, since N is $\mathfrak{F}^*$-central in H. But then $G \in \mathcal{Y}_{\mathfrak{F}}$, a contradiction. Therefore, $N = C_G(N)$ and so N is complemented in G by M, say. Now, since N is covered by F and is a self-centralizing chief factor of F, it follows that $F \cap M \in \mathfrak{F}(p)$. By [1, Theorem 5.12] we can assume that $F \cap M$ is an $\mathfrak{F}$-projector of M. Hence, $M \in \alpha(p) \subseteq \mathfrak{F}(p)$ so that $G \in S_p\mathfrak{F}(p) = \mathfrak{F}(p) \subseteq \mathfrak{F} \subseteq \mathcal{Y}_{\mathfrak{F}}$, a contradiction. This completes the proof.

The saturated formation $\mathfrak{F}$ is said to satisfy *condition* C provided that if G is a primitive group with unique minimal normal subgroup N, and F is an $\mathfrak{F}$-projector of G with unique minimal normal subgroup N, then $F = G$.

Theorem 4. *$\mathfrak{F}$ is $\mathcal{C}$-special if and only if $\mathfrak{F}$ satisfies condition* C.

Proof. Assume that $\mathfrak{F}$ satisfies condition C and let $\{1\} \neq G \in \alpha(p)$. Let V be a faithful, irreducible $Z_p[G]$-module such that V_F, F an $\mathfrak{F}$-projector of G, is an irreducible $Z_p[F]$-module. Let $H = [V]G$ be the semidirect product of V by G. By Doerk [2, Hilfssatz 2.6], $\bar{F} = VF$ is an $\mathfrak{F}$-projector of H. Since H is a primitive group and V_F is irreducible, it follows that $G = F \in \mathfrak{F}(p)$. Hence, $\mathfrak{F}$ is $\mathcal{C}$-special by Theorem 3.

Conversely, assume that $\mathfrak{F}$ is $\mathcal{C}$-special. Let G be a primitive group with unique minimal normal subgroup N and let M complement N in G. Assume that the $\mathfrak{F}$-projector F of G covers N and let $|N| = p^n$, $n > 0$. Because of [1, Theorem 5.12] we can assume $F \cap M$ is an $\mathfrak{F}$-projector of M. Suppose that N is the unique minimal normal subgroup of F. Then $M \in \alpha(p) \subseteq \mathfrak{F}(p)$ because of Theorem 3, hence $G = [N]M \in S_p\mathfrak{F}(p) \subseteq \mathfrak{F}$. This completes the proof.

The saturated formation $\mathfrak{F}$ is said to satisfy *condition* A (cf. Huppert [7, p. 569]) if there exists a formation $\mathfrak{X}$ such that $\mathfrak{F}(p) = S_p\mathfrak{X}$ for each prime p. Further, $\mathfrak{F}$ is said to satisfy *condition* B (cf. Huppert [7, p. 569]) if $\mathfrak{F}$ is subgroup-inherited and if $G \in \mathfrak{F}$ and N is a minimal normal subgroup of G, then Aut $(N) \in \mathfrak{F}$.

Proposition 5. *If $\mathfrak{F}$ satisfies either condition* A *or condition* B, *then $\mathfrak{F}$ is $\mathcal{C}$-special.*

Proof. Because of Theorem 4 it is enough to show that $\mathfrak{F}$ satisfies condition C. Let G be a primitive group with unique minimal normal subgroup N, and let F be an $\mathfrak{F}$-projector of G which contains N as its unique minimal normal subgroup.

Assume that $\mathfrak{F}$ satisfies condition B. Then $G/N \in \mathfrak{F}$, hence $G = F$ and $\mathfrak{F}$ satisfies condition C.

Next assume that $\mathfrak{F}$ satisfies condition A and let $\mathfrak{X}$ be a formation such that $\mathfrak{F}(p) = S_p\mathfrak{X}$ for each p. Assume that $G^{\mathfrak{F}} \neq \{1\}$. If $N = G^{\mathfrak{F}}$, then $G = F$, a contradiction. Hence, N is a proper subgroup of $G^{\mathfrak{F}}$. Let N be a power of the prime p. Then

$$G/G^{\mathfrak{F}} \cong F/F \cap G^{\mathfrak{F}} \cong (F/N)/(F \cap G^{\mathfrak{F}}/N) \in \mathfrak{X} \subseteq \mathfrak{F}(p).$$

Let $G^{\mathfrak{F}}/K$ be a q-chief factor of G. Since $G^{\mathfrak{F}} \subseteq C_G(G^{\mathfrak{F}}/K)$, it follows that $G/C_G(G^{\mathfrak{F}}/K) \in \mathfrak{X}$ and so $G^{\mathfrak{F}}/K$ is an $\mathfrak{F}$-central chief factor of G. This contradiction completes the proof.

Let $\mathfrak{N}^{(0)}$ denote the class of groups consisting of the group of order 1, and for each positive integer i, let $\mathfrak{N}^{(i)}$ denote the formation of all groups G of Fitting length at most i. Then, for $i \geq 1$, $\mathfrak{N}^{(i)}$ has a canonical definition $\{S_p\mathfrak{N}^{(i-1)}\}$. Hence, the saturated formations $\mathfrak{N}^{(i)}$, $i \geq 1$, satisfy condition A and so are $\mathcal{C}$-special. The saturated formation $\mathfrak{X}$ of supersoluble groups satisfies condition B, hence $\mathfrak{X}$ is $\mathcal{C}$-special. Saturated formations $\mathfrak{F}$ satisfying condition B are easily classified. This will be the content of a later paper by the first author.

3. $\mathcal{C}$-special formations for which $\mathcal{C}_{\mathfrak{F}}$ is saturated. In this section, we show

Theorem 6. *$\mathcal{C}_{\mathfrak{F}}$ is saturated if and only if $\mathfrak{F} = S$, the class of all finite soluble groups.*

In order to prove this theorem we need the following lemma.

Lemma 7. *If $\mathcal{C}_{\mathfrak{F}}$ is locally defined by the family $\{\mathcal{C}_{\mathfrak{F}}(p)\}$, one for each prime p, of full and integrated homomorphs then $\mathcal{C}_{\mathfrak{F}} = \mathfrak{N}\mathfrak{F}$. In particular, $\mathfrak{F}$ is $\mathcal{C}$-special.*

Proof. Since $\mathfrak{N}\mathfrak{F} \subseteq \mathcal{C}_{\mathfrak{F}}$, $S_p\mathfrak{F} \subseteq \mathcal{C}_{\mathfrak{F}}(p)$ for each prime p. Hence, it suffices to show that $\mathcal{C}_{\mathfrak{F}}(p) \subseteq S_p\mathfrak{F}$ for each prime p. Suppose that there is a prime p such that $\mathcal{C}_{\mathfrak{F}}(p) \nsubseteq S_p\mathfrak{F}$. Let $G \in \mathcal{C}_{\mathfrak{F}}(p)\backslash S_p\mathfrak{F}$ be of minimal order. Since $\mathcal{C}_{\mathfrak{F}}(p)$ is a homomorph and $S_p\mathfrak{F}$ is a saturated formation, G has a unique minimal normal subgroup N which is self-centralizing in G. Let $|N| = q^n$, $n > 0$. Clearly $q \neq p$. Let F be an $\mathfrak{F}$-projector of G and assume first that F avoids N. Let M be a complement of N in G which contains F. By Doerk [2, Hilfssatz 1.2], G has a faithful, irreducible $Z_p[G]$-module V such that V_M contains the irreducible, trivial $Z_p[M]$-module as a factor module. In particular, V_F is not an irreducible $Z_p[F]$-module. Let $H = [V]G$. Since $G \in \mathcal{C}_{\mathfrak{F}}(p) \subseteq \mathcal{C}_{\mathfrak{F}}$, we have $H \in S_p\mathcal{C}_{\mathfrak{F}}(p) = \mathcal{C}_{\mathfrak{F}}(p) \subseteq \mathcal{C}_{\mathfrak{F}}$. Thus, since $C_V(F^{\mathfrak{F}(p)}) \neq \{1\}$, VF is an $\mathfrak{F}$-projector of H. However, V is not a chief factor of VF, though it is a chief factor of H, contrary to $H \in \mathcal{C}_{\mathfrak{F}}$.

Assume next that F covers N. In particular, since N is self-centralizing in G, $F \in \mathfrak{F}(q)$. For the same reason, $G \in \mathcal{C}_{\mathfrak{F}}(q)$ since $G \in \mathcal{C}_{\mathfrak{F}}$ and therefore $M \in \mathcal{C}_{\mathfrak{F}}(q)$. Moreover, since $M \in S_p\mathfrak{F}$, we have $|M: M \cap F| = p^{\alpha}$, $\alpha > 0$, and so by Doerk [2, Hilfssatz 1.2], M has a nontrivial irreducible $Z_q[M]$-module W such that $W_{F \cap M}$ contains the trivial $Z_q[F \cap M]$-module as a factor module. Since $F \cap M$ is an abnormal subgroup of M, certainly $\dim_{Z_q} W \neq 1$. Thus $W_{F \cap M}$ is not an irreducible $Z_q[F \cap M]$-module. Now, let $K = [W]M$, the semidirect product of W by M. Since $F \cap M \in \mathfrak{F}(q)$, it follows by Doerk [2, Hilfssatz 2.6], that $\overline{F} = [W](F \cap M)$ is an $\mathfrak{F}$-projector of K. Also, since $M \in \mathcal{C}_{\mathfrak{F}}(q)$, $K \in \mathcal{C}_{\mathfrak{F}}(q) \subseteq \mathcal{C}_{\mathfrak{F}}$. However, W is not a chief factor of $\overline{F}$, a contradiction. Hence, the lemma follows.

By an argument similar to the above, one can prove

Theorem 8. *If $\mathfrak{X}$ is a saturated formation and $\mathfrak{N}\mathfrak{F} \subseteq \mathfrak{X} \subseteq \mathcal{C}_{\mathfrak{F}}$, then $\mathfrak{N}\mathfrak{F} = \mathfrak{X}$.*

In view of Lemma 7 and Theorem 1.3 of Doerk and Hawkes [3] we have

Corollary 9. *If $\mathcal{C}_{\mathfrak{F}}$ is saturated; then $\mathfrak{F} = S_{p'}\mathfrak{F}(p)$ for some prime p.*

We now prove Theorem 6.

Proof of Theorem 6. Suppose that $\mathcal{C}_{\mathfrak{F}}$ is saturated. In order to show that $\mathfrak{F} = S$, it will be sufficient to show that $\mathfrak{F} = \mathfrak{F}(p)$ for each prime p. Suppose to the contrary that $\mathfrak{F} = \mathfrak{F}(p)$ for some prime p and let $G \in \mathfrak{F}\backslash\mathfrak{F}(p)$ be of minimal order. Then G contains a unique minimal normal subgroup N. Let $|N| = t^{\alpha}$, t a prime and $\alpha > 0$. Since $\mathfrak{F}(p) = S_p\mathfrak{F}(p)$, $t \neq p$.

Next, let K be an isomorphic copy of the group G in Huppert [9, Beispiel 2.9], if p is odd; otherwise, let K be the semidirect product of the natural 2-dimensional GF(5)-module of GL(2, 5) by the normalizer in SL(2, 5) of a Sylow 2-subgroup of SL(2, 5). In either case, K has a unique minimal normal subgroup M which is self-centralizing in K and which is also a minimal normal subgroup of a Hall p'-subgroup K^p of K. Moreover, if $L/M = O_p(K/M)$, then K/L is a p'-group.

Consider now the wreath product $H = K \wr G = [K_1 \times \cdots \times K_n]G$ which is defined by a transitive permutation representation of G of degree $n > 1$. Since $K_1 \times \cdots \times K_n/L_1 \times \cdots \times L_n$ is a p'-group and $G \in \mathfrak{F}$, it follows from Corollary 9 that $H/L_1 \times \cdots \times L_n \in \mathfrak{F}$. Thus $H/M_1 \times \cdots \times M_n \in \mathfrak{N}\mathfrak{F} \subseteq \mathcal{C}_{\mathfrak{F}}$. Also, since M is an eccentric chief factor of K^p, it follows, by Hawkes [5, Lemma], that $M_1 \times \cdots \times M_n$ is a chief factor of $K^p \wr G = [K_1^p \times \cdots \times K_n^p]G$. But, by Carter and Hawkes [1, Theorem 5.12] applied to $H/M_1 \times \cdots \times M_n$ and by another application of Corollary 9, $[K_1^p \times \cdots \times K_n^p]G$ is a subgroup of some $\mathfrak{F}$-projector F of H. Hence, F satisfies the hypothesis of Lemma 2, and so $H \in \mathcal{C}_{\mathfrak{F}}$. However, $H \in \mathfrak{N}\mathfrak{F}$, whereas $\mathfrak{N}\mathfrak{F} = \mathcal{C}_{\mathfrak{F}}$, by Lemma 7. Thus we have a contradiction. Hence, $\mathfrak{F} = \mathfrak{F}(p)$ for each prime p, and so $\mathfrak{F} = S$. Since the converse of the theorem is trivially true, the proof is complete.

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